Structural Stability and Bifurcations
a Quick Trip

The Stability Dogma: Since in any physical model the parameters and constants are only known approximately, the most important dynamical systems are those that are stable under perturbation.

[This is meant to be a bit ironic]

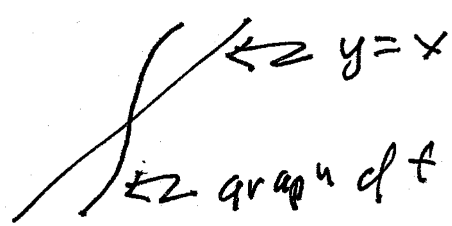
- So what does it mean for an entire dynamical system to be stable under perturbation?
  - Nearby systems should be topologically conjugate to the original.
  - What does nearby mean?
  - Topology on functions:
    say M is a compact domain in $\mathbb{R}^n$ and $f, g : M \to M$ are C^r (i.e. have r continuous derivatives)
  - The C^r-metric is

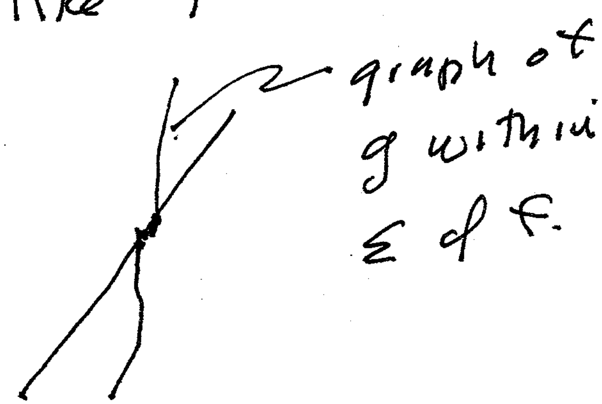
$$\underline{d_r(f,g)} = \max_{x \in M} \| f(x) - g(x) \| + \| Df(x) - Dg(x) \| + \cdots + \| D^r f(x) - D^r g(x) \|$$

[also works on manifolds using local charts]

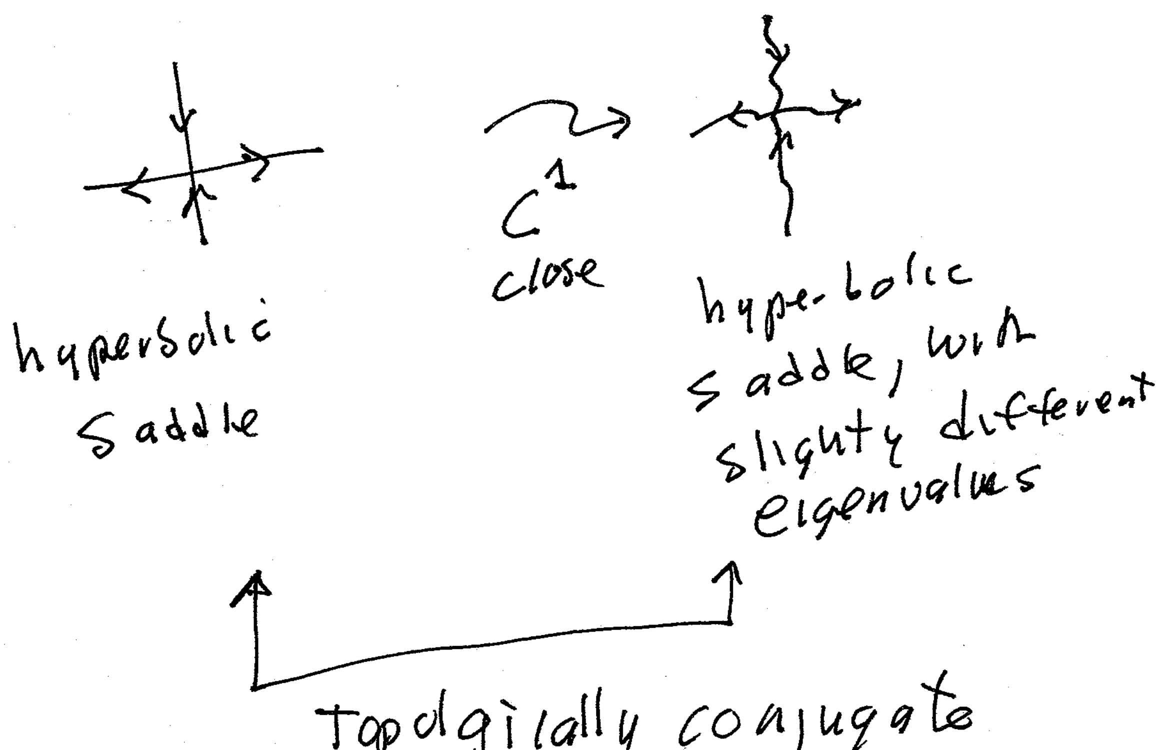
<u>Def</u> $f : M \to M$ is <u>C^r-structurally stable</u> if $\exists\ \varepsilon > 0$ so that $d_r(f,g) < \varepsilon$ implies g is <u>topologically</u> conjugate to f

<u>Note</u>: C^0-structural stability is not interesting; you can make f and g equal outside an ε-ball and inside do whatever you like dynamically





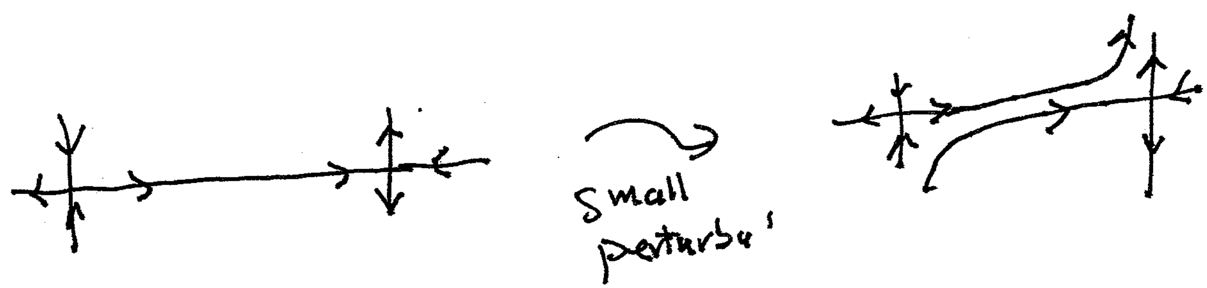
<u>NOTE</u>: Note that even though we require f and g to have nearby derivatives the conjugacy is only a homeomorphism (smooth conjugacy conjugacy is called rigidity). Smooth conjugacy preserves, for example, eigenvalues at fixed points so is much too strong



. The most studied is C^1 structural stability

Which systems are C^1- structurally stable?

* <u>Morse - Smale</u> – Only recurrent points are hyperbolic periodic points with no connections



* Hyperbolic attractors – Solenoid and Pyklin

* Anosov diffeos – cat map on $\mathbb{T}^2$

* Axiom A ....

Parameterized families

- From physical problems one is often given a family $f_\eta : M \to M$ depending on a parameter η
- So it is natural to study the dynamics a η - changes.
- For simplicity, assume $\eta \in [0,1]$

  and f smoothly depends on both x and \eta
- If η_0 is such that f_{η_0} is structually stable $\Rightarrow \exists\ \varepsilon$ so that $\eta \in (\eta_0 - \varepsilon, \eta_0 + \varepsilon)$ implies f_η is topologically conjugate to f_{η_0}

- If η_0 is such that f_{η_0} is <u>not</u> structurally stable, the η_0 is called a <u>bifurcation point</u>

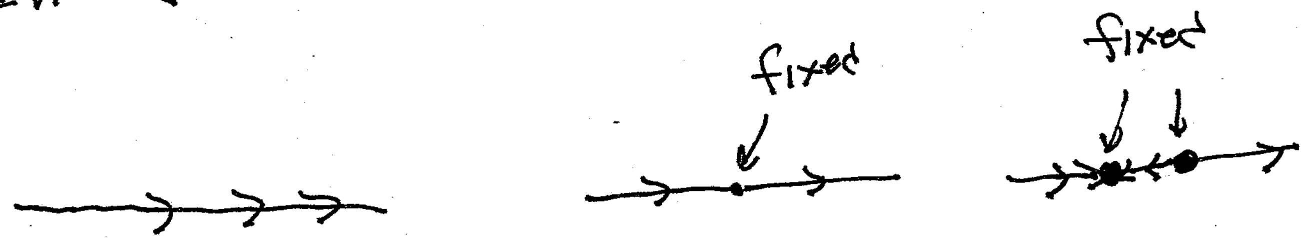
- As bifurcation points dynamics is born or dies as η is increased through η_0

- While very complicated things can happen at bifurcations, there is a list of the simplest, most common bifurcations

- We look at 3 of them.

The Saddle node

- In 1 dim

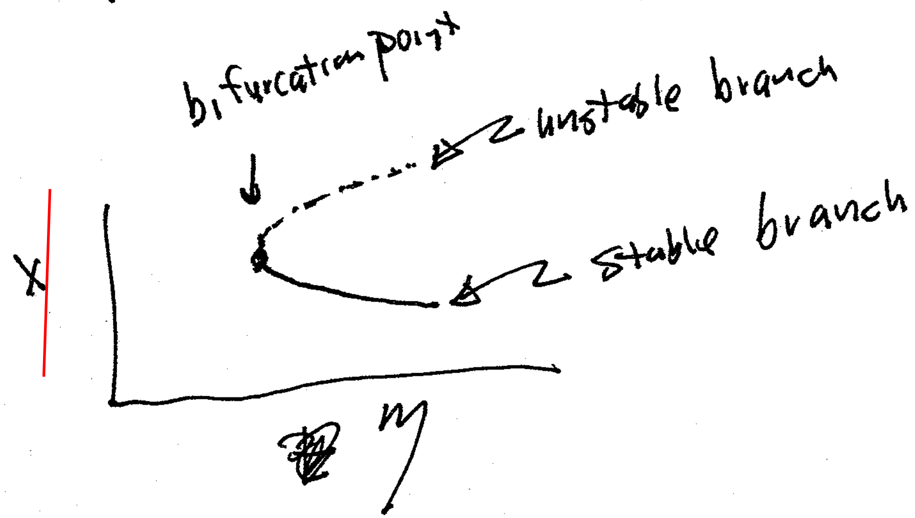


push to the left creates fixed point

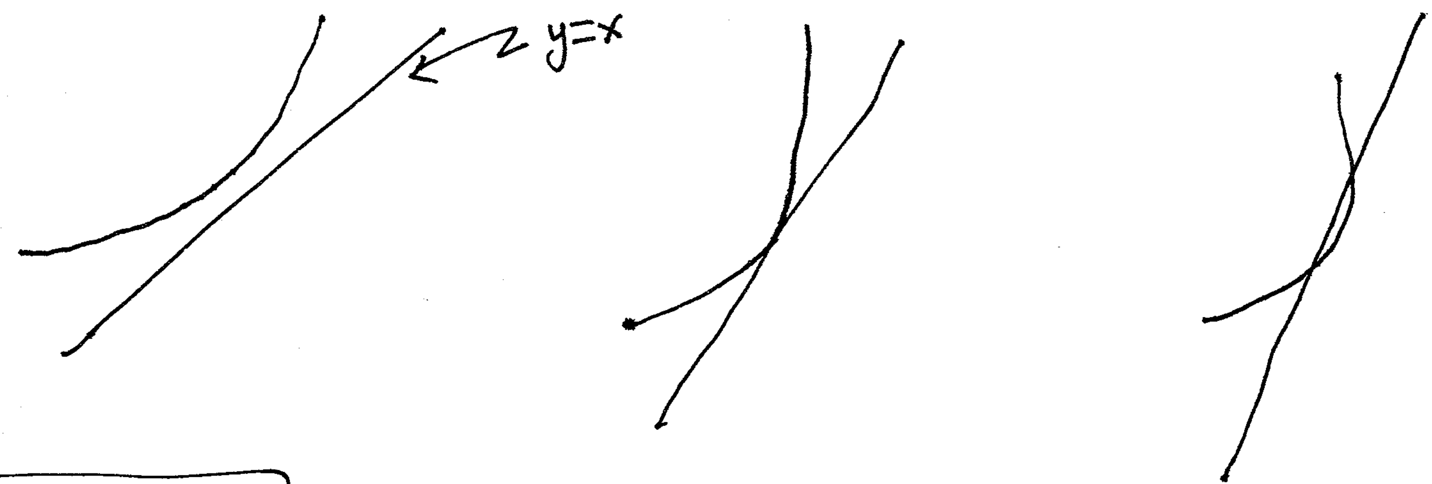
further push creates sink-source pair

Viewing right to left a sink-source pair annihilate each othe

Bifurcation diagram



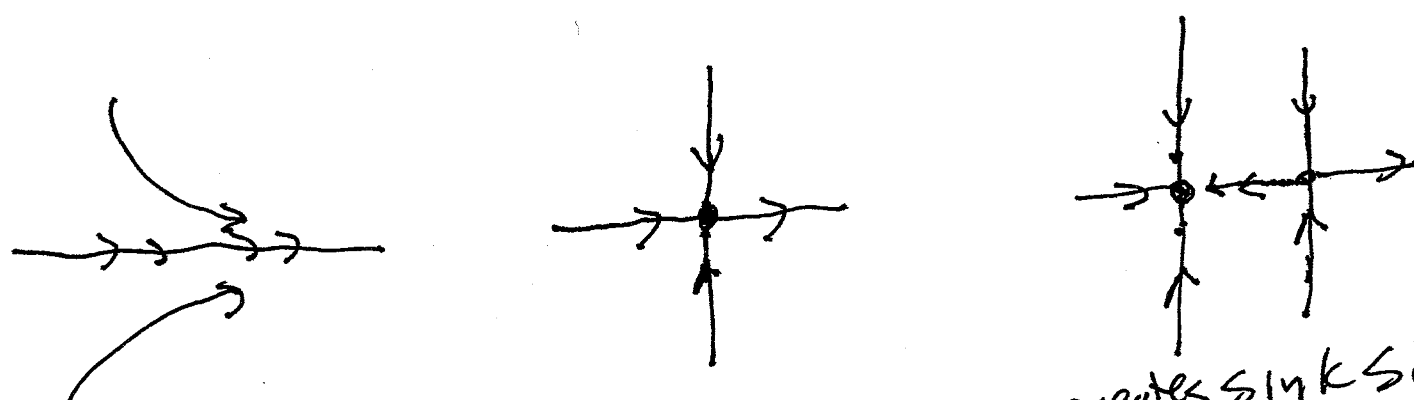
graph



$\boxed{f'(p)=1 \text{ at bifurcation}}$

bifurcation point

In higher dimensions, the addition directions don't change their stability

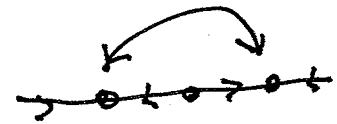


creates sink saddle pair.

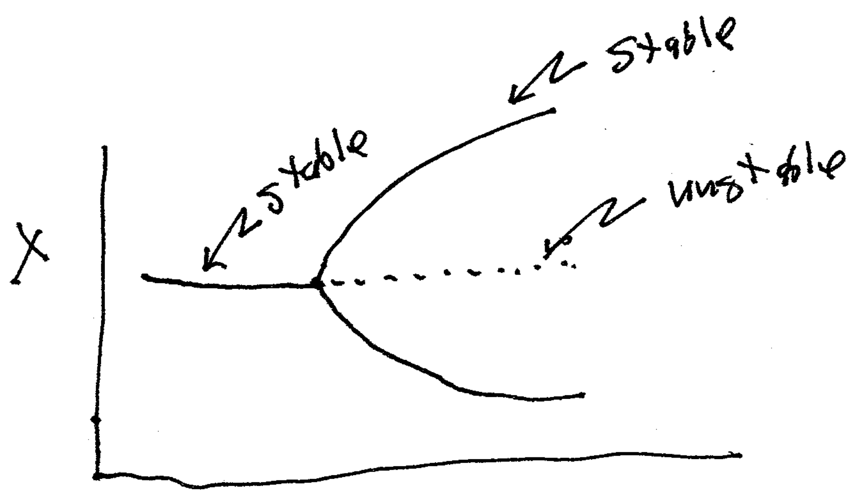
$\boxed{\text{Period doubling}}$ In 1 dim

flip attractor

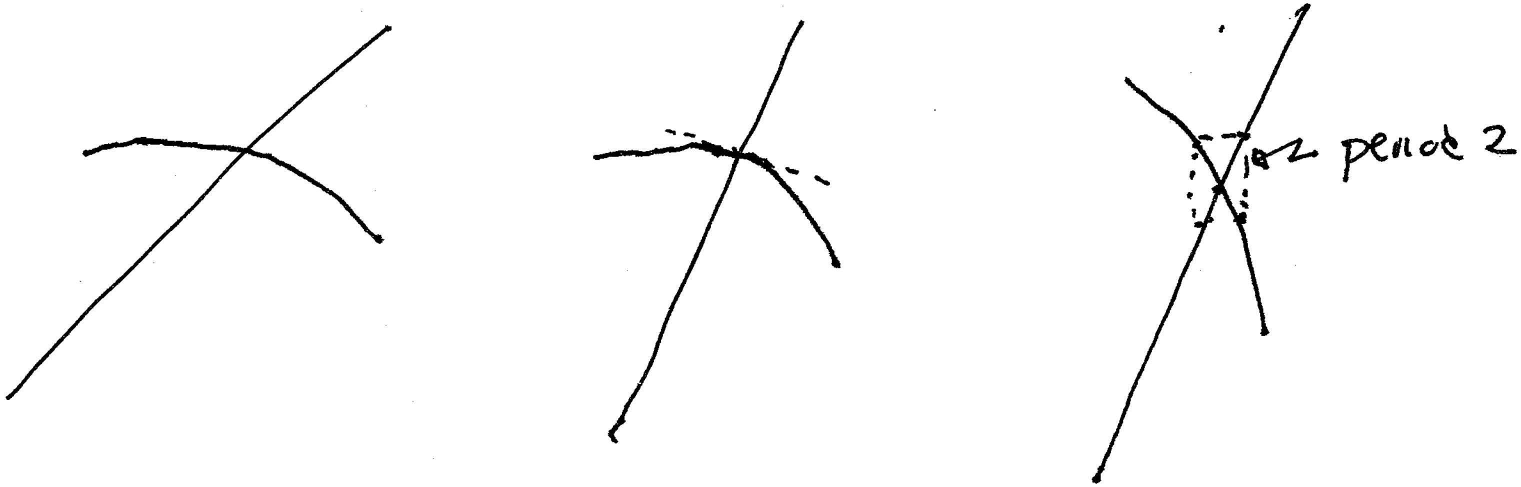
push out from fixed point yields $f'(p) = -1$ at fixed pt



attracting period 2 and fixed point has become unstable



graphs

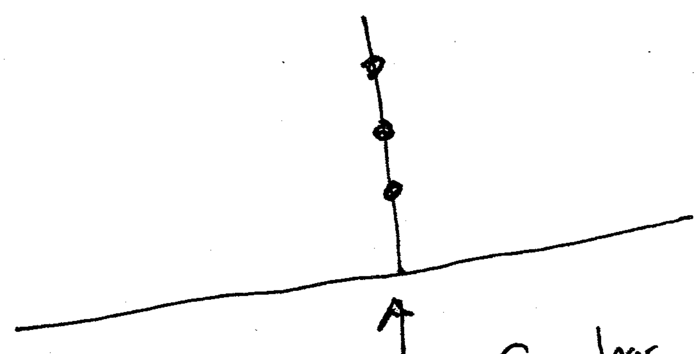


As with saddle node, this happens in higher dimensions with all but one direction not changing stability

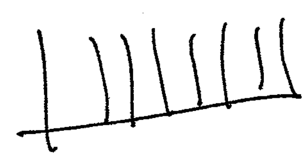
Bifurcation diagram of quadratic family

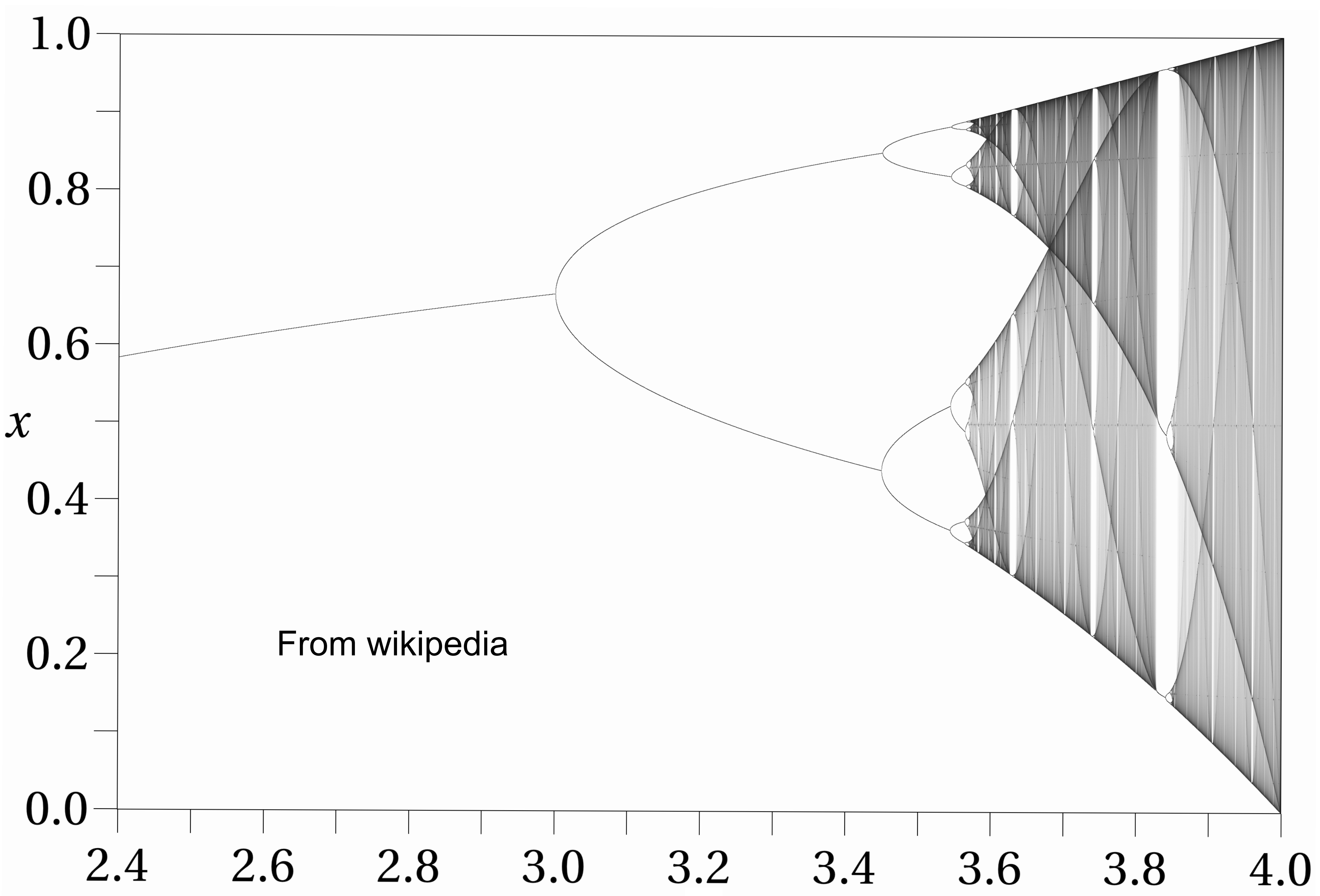
Shows doubling, saddle nodes (and many others not discussed) $f_\mu = \mu x(1-x)$ 

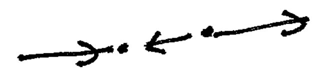
Algrorithm: Fix μ, pick random initial point x_0, iterate 100 times, start plotting $f_\mu^{100+1}(x_0)$, ... etc mark the iterates as a dot, in that segment



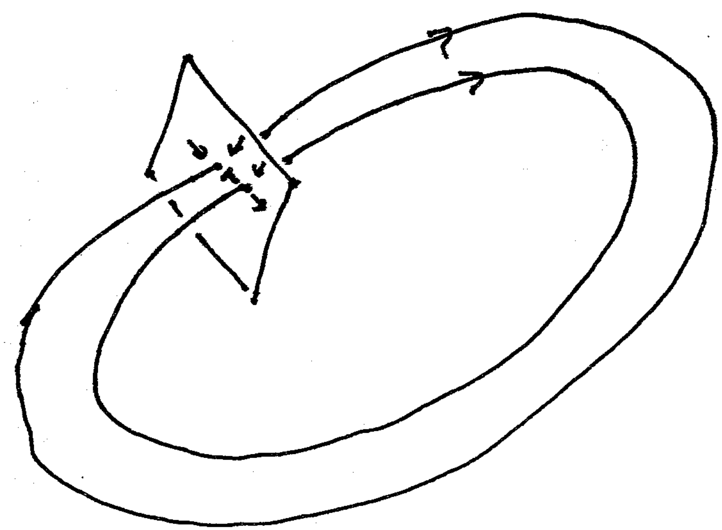
μ f_μ has an attracting period 3

do for lots of slices 

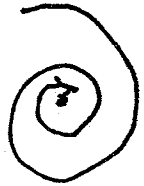

1.0
0.8
0.6
x
0.4
0.2
0.0
2.4
2.6
2.8
3.0
3.2
3.4
3.6
3.8
4.0
From wikipedia

Saddle nodes can happen for rest points of flows 

or in the Poincaré sections of periodic orbits

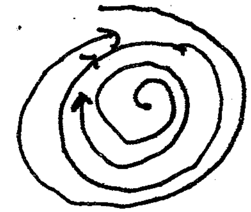


Hopf Bifurcation - For flows



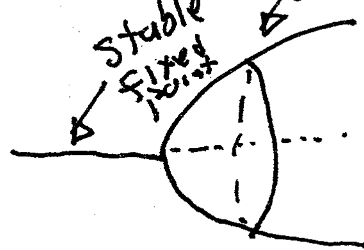
attracting Spiral

push out so pure rotation in linearization

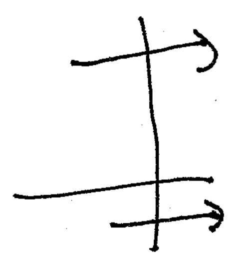


stable Circle. = limit cycle.

Bifurcation diagram



eigen values at p



- conjugate pair cross the imaginary axis at the bifurcation point.

- For each bifurcation there is a list of conditions that f_μ must satisfy in order to have a bifurcation of the given type

- There are many more bifurcations.....

video shown is on YouTube, AppDynSys series by Prof. Ghrist.