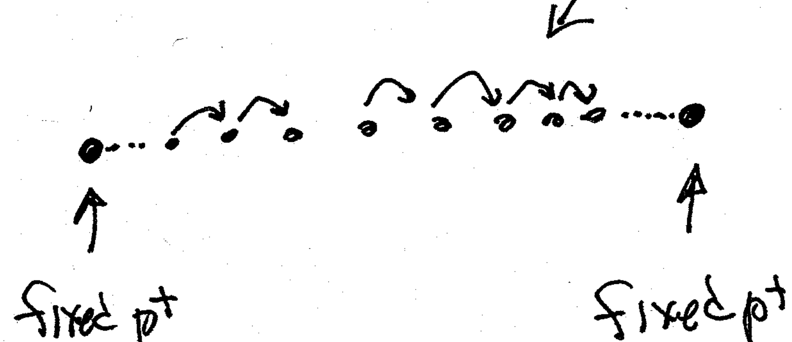
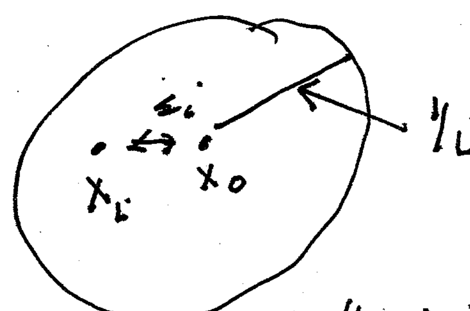
- There are some somewhat technical issues regarding full orbits $o(x,h) = \{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots\}$ and partial orbits $o^+(x,h) = \{x, f(x), f^2(x), \ldots\}$ (the forward orbit) and $o^-(x,h) = \{\ldots, f^{-2}(x), f^{-1}(x), x\}$.
- Here's one: If (h, X) is transitive then by definition there is a full orbit that is dense $\overline{o(x,h)} = X$. Is there a forward orbit that is dense?
- In general, no: the central full orbit is dense, but there is no dense forward orbit.



- *Theorem*: $h : X \circlearrowleft$ is a homeomorphism of a compact metric space. If h has a dense full orbit it has a dense forward orbit as long as X has no ~~isolated~~ isolated points.

PROOF: Let x_0 be such that $\overline{o(x_0, h)} = X$. We claim $\exists |n_i| \to \infty$ with $h^{n_i}(x_0) \to x_0$. To prove this note that since x_0 is not isolated, there exists $x_i \neq x_0$ with $x_i \in B_{1/i}(x_0)$. Let $\varepsilon_i = d(x_0, x_i)$



Since $o(x_0, h)$ is dense, $\exists n_i$ with $d(h^{n_i}(x_0), x_i) < \min\{1/i, \varepsilon_i\}$ and so $h^{n_i}(x_0) \neq x_0$ and $d(h^{n_i}(x_0), x_0) < 1/i$. Since $1/i \to 0$ there are infinitely many distinct n_i and $h^{n_i}(x_0) \to x_0$ and $|n_i| \to \infty$ proving the claim.

Now note that $h^{n_i}(x_0) \to x_0$ implies $h^{n_i+k}(x_0) \to h^k(x_0)$
for any integer k.

e infinitely many distinct n_i, either

(a) infinitely many of them are >0 or
(b) " " " " " <0.

In case (a), since for any k, n_i+k is eventually >0 we have

$$O(x_0,h) \subseteq \overline{O^+(x_0,h)}$$

In case (b), similarly $O(x_0,h) \subseteq \overline{O^-(x_0,h)}$

In case (a) $X = \overline{O(x_0,h)} = \overline{\overline{O^+(x_0,h)}} = \overline{O^+(x_0,h)}$ and we are done.

In case (b) we first claim that given any open U, V there is an $n<0$ with $h^n(U) \cap V \neq \phi$

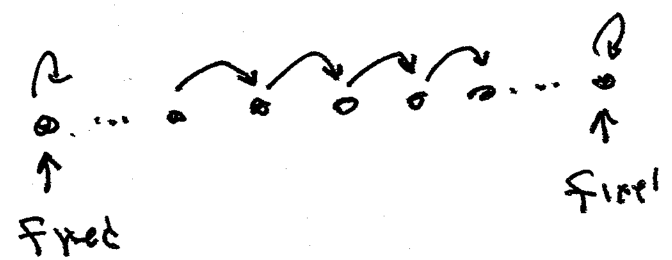
To prove this claim, note that using a proof similar to that above, since $O(x_0,h) \subseteq O^-(x_0,h)$ for any open U there are arbitrarly negative k (i.e. $k<0$ and $|k|$ large) with $h^k(x_0) \in U$. In particular given $l<k<0$ with $h^k(x_0) \in U$ and $h^l(x_0) \in V$ and so $x_0 \in h^{-k}(U) \cap h^{-l}(V)$ so $h^{l}(x_0) \in h^{l-k}(U) \cap V$ with $l-k<0$, proving the claim.

Now we proceed with a Baire argument as in the Transitivity Theorem. Let $\{U_i\}$ be a countable base for the topology and $W_i = \{x : \exists\, n>0 \text{ with } h^n(x) \in U_i\} = \bigcup_{n=0}^{\infty} h^{-n}(U_i)$

By the claim, each W_i is dense since it hits every open set. And it is open as union of open sets.

Thus By the Baire Category Theorem, $\bigcap W_i$ is dense, G_δ and consists of all x with $O^+(x,h)$ dense, so there is at least one! such x. $\blacksquare$

NOTE on ~~Pr~~ problem 9 in HW: without the hypothesis of no isolated points the example above



has a dense full orbit and a non-constant Lyapunov function.

Symbolic Dynamics

$$\Sigma_2^+ = \{0,1\}^{\mathbb{N}} = \{ .s_0 s_1 s_2 \dots : s_i \in \{0,1\} \}$$ onesided two shift

$$\Sigma_2 = \{0,1\}^{\mathbb{Z}} = \{ \dots s_{-2} s_{-1} . s_0 s_1 s_2 \dots : s_i \in \{0,1\} \}$$ full two-shift

The "decimal point" is a notational convenience to keep track of the zero position

eg $\underline{s} \in \Sigma_2^+$ $\quad \underline{s} = .0110110001001 11 \dots$

$\underline{s} \in \Sigma_2$ $\quad \underline{s} = \dots\dots 011011.110011011 \dots\dots$

For $\underline{s}, \underline{t} \in \Sigma_2^+$ define

$$d(\underline{s}, \underline{t}) = \sum_{i=0}^{\infty} \frac{|s_i - t_i|}{2^i}$$

since $|s_i - t_i| \leq 1$, sum conv. by Weierstrass M-test

and $d(\underline{s}, \underline{t}) \leq \sum_{i=0}^{\infty} \frac{1}{2^i} = \frac{1}{1-\frac{1}{2}} = 2$

<u>Lemma</u>: This yields a metric on Σ_2^+

For $\underline{s}, \underline{t} \in \Sigma_2$ define

$$d(\underline{s}, \underline{t}) = \sum_{i \in \mathbb{Z}} \frac{|s_i - t_i|}{2^{|i|}}$$

which is also a metric on Σ_2

When are two sequences close?

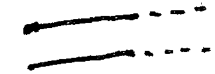
In Σ_2^+

Lemma (a) If $s_i = t_i$ for $i = 0, \dots, n \Rightarrow d(\underline{s}, \underline{t}) \leq 1/2^n$

(b) If $d(\underline{s}, \underline{t}) < \frac{1}{2^n} \Rightarrow s_i = t_i$ for $i = 0, \dots, n$

Proof: (a)
$$d(\underline{s}, \underline{t}) = \sum_{i=0}^{n} \frac{|s_i - t_i|}{2^i} + \sum_{i=n+1}^{\infty} \frac{|s_i - t_i|}{2^i}$$
$$\leq 0 + \sum_{i=n+1}^{\infty} 1/2^i = 1/2^n$$

(b) The contrapositive. Say $s_j \neq t_j$ for some $j \leq n$
$$\Rightarrow d(\underline{s}, \underline{t}) \geq \frac{1}{2^j} \geq \frac{1}{2^n}.$$

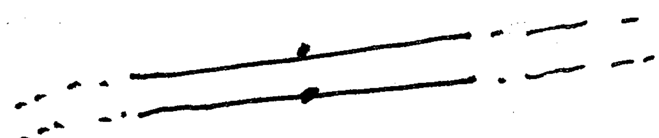


- So sequences are close when they agree for a long time. Sometimes a different metric is defined

$$\rho(\underline{s}, \underline{t}) = \frac{1}{n} \quad \text{when } s_i = t_i \text{ for } i = 0, \ldots, n-1 \text{ and } s_n \neq t_n$$

It defines the same topology on Σ_2^+.

Similarly, $\underline{s}, \underline{t} \in \Sigma_2$ are close if $s_i = t_i$ for $|i| \leq n$ for large n



Dynamics

- The left s... map σ on Σ_2^+ is

$$\sigma(.s_0 s_1 \ldots) = .s_1 s_2 s_3 \ldots$$

so it is 2-1

- The left s . map σ on Σ_2 is

$$\sigma(\ldots s_{-2} s_{-1} . s_0 s_1 s_2 \ldots) = \ldots s_{-2} s_{-1} s_0 . s_1 s_2 \ldots$$

- <u>Lemma</u>: σ on Σ_2^+ is continuous, and σ on Σ_2 is a homeomorphism

<u>Theorem</u> σ on Σ_2 and σ on Σ_2^+ are transitive i.e. have dense orbits

<u>Proof</u> We do σ on Σ_2^+, the other case is similar

We make a list of all the blocks i.e. finite pieces of symbols

$b_1 = 0$
$b_2 = 1$
$b_3 = 00$
$b_4 = 01$
$b_5 = 10$
$b_6 = 11$
$b_7 = 000$
$b_8 = 001$
$\vdots$

Form the sequence

$\underline{t} = b_1 b_2 b_3 b_4 \ldots$

$= 0 1 00 01 10 11 000 001 \ldots$

and Given $\varepsilon > 0$ and a sequences $\underline{s}$ we must find an k so that

$d(\sigma^k(\underline{t}), \underline{s}) < \varepsilon$

Let n be such that $\frac{1}{2^n} < \varepsilon$ and

~~assume~~ let b_j be the block so that

$(b_j)_i = t_i$ for $i = 0, \ldots, n$

and chose k so that $\sigma^k(\underline{t}) = b_j \cdot b_{j+1} \cdots$

$\Rightarrow$ & since $\underline{s}$ and $\sigma^k(\underline{t})$ agree in the first

n slots $d(\underline{s}, \sigma^k(\underline{t})) \leq \frac{1}{2^n} < \varepsilon$ $\blacksquare$