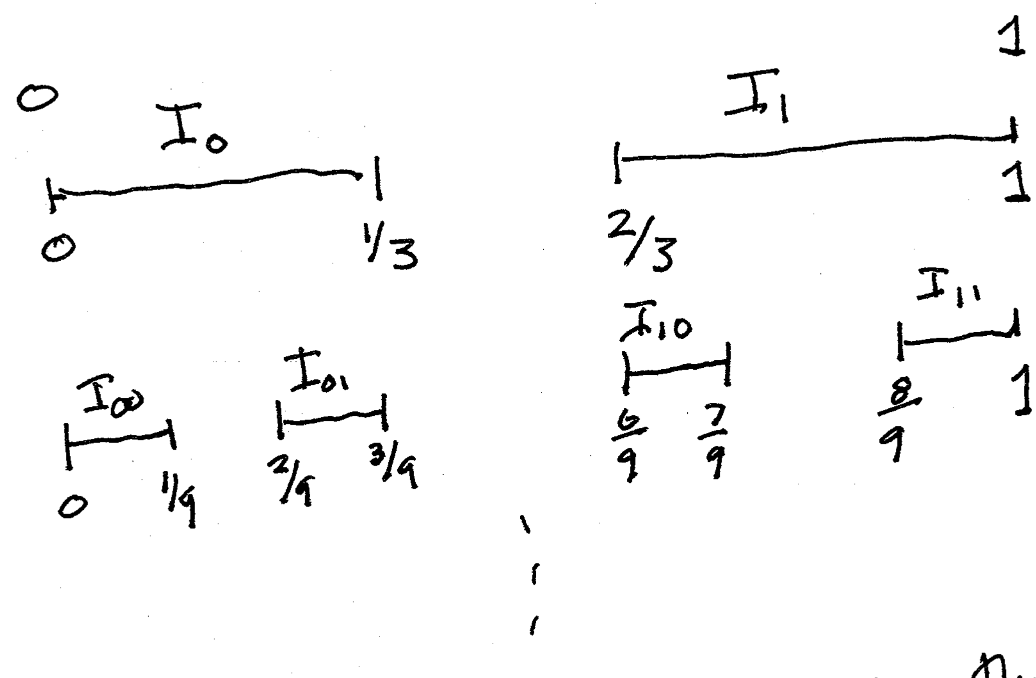
The Cantor Middle Third Set

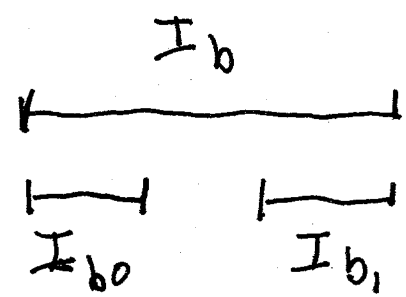


$I_0 \cup I_1 = S_1$

$I_{00} \cup I_{01} \cup I_{10} \cup I_{11}$
$= S_2$

at each step delete the middle third of I_b to get a left interval I_{b0} and a right interval I_{b1}

b is a block of 0's and 1's.



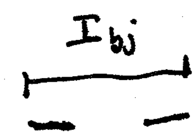
S_n = union of intervals on the n^{th} level.

The Cantor middle third is

$$C = \bigcap_{n=1}^{\infty} S_n$$

(1) Since each S_n is compact and nonempty and $S_{n+1} \subseteq S_n$, C is compact and nonempty

(2) C contains no intervals: Proof: Say $J \subseteq C$ so $\exists\ x \in J \subseteq C$. Now by construction any x is formed by $\bigcap I_{b_j}$ where $|b_j| = j$
$b_{j+1} = b_j$ followed by 0 or 1. But the lengths $|I_{b_j}| = 1/2^j$ and so $|I_{b_j}| \to 0$ so for some j, $I_{b_j} \subseteq J$. But then



since I_{b_j} contains points not in C, contradiction

Since $C \subseteq [0,1]$ this means that C is nowhere dense and also completely disconnected (C connected components are all points).

(3) C is perfect (every point is a limit point).

Pick $x \in C$ and as in (2) write $x = \bigcap I_{b_j}$.

Given $\varepsilon > 0$ pick N so that $1/2^N < \varepsilon$

and ~~let~~ define $b_j' = b_j$ when $1 \le j \le N$

and $b_j' = b_N \underbrace{000 \cdots 0}_{N-j \text{ zeros}}$ when $j \ge N$ then

$y = \bigcap I_{b_j'} \in C$ and $d(x,y) < \frac{1}{2^N} < \varepsilon$

$\boxed{\text{What Does this have to do with } \Sigma_2^+ \text{?}}$

- Given a sequense $\underline{s} \in \Sigma_2^+$ let its initial length $n+1$-block be $\underline{s}(0:n)$ [matlab notation?]

- **Theorem**: The map $\varphi : \Sigma_2^+ \to C$ given by $\varphi(\underline{s}) = \bigcap_{j=0}^{\infty} I_{\underline{s}(0:j)}$ is a homeomorphism.

Proof Since any point $x \in C$ is represented uniquely as $x = \bigcap_{j=1}^{\infty} I_{b_j}$, φ is bijective. And a sequence $\underline{s}^{(i)} \to \underline{s}$, if ~~$\underline{s}^{(i)}(0:i) = \underline{s}(0:i)$ for large i~~ in Σ_2^+, and only if $\forall n\ \exists J$ so that $\underline{s}^{(j)}(0:n) = \underline{s}(0,n)\ \forall j \geq J$

And if $x^{(j)} = \bigcap_{n=0}^{\infty} I_{b^{(j)}_n}$ and $x = \bigcap_{n=0}^{\infty} I_{b_n}$

then $x^{(j)} \to x$ if and only if $\forall n \ \exists J$

So that $b_n^{(j)} = b_n \quad \forall j \geq J$

Thus $\underline{s}^{(j)} \to s \iff \varphi(\underline{s}^{(j)}) \to \varphi(s)$

So φ and φ^{-1} are continuous •

<u>CORR</u>: C is uncountable

<u>Remark</u>: (1) A Cantor space is any non-empty, compact, totally disconnected, metrizable space. Th'm: They are a homeomorphic to the Cantor Middle Third.

(2) C can also be realized as the collection of base 3 expansions of numbers in $[0,1]$ which contain no 1. With this representation the map is, for example, $\underline{s} = .011001.. \in \Sigma_2^+$ maps to $x = .022002..$ is base 3. OR explicitly,

$$\varphi(\underline{s}) = \sum_{n=0}^{\infty} \frac{2 s_n}{3^{n+1}}$$

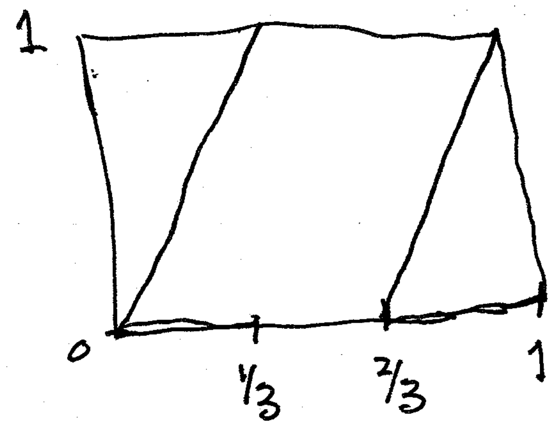
(3) By varying the ~~pr~~ proportion taken out at each level one can create Cantor spaces $C' \subseteq [0,1]$ with positive Lebesgue measure and has so $U = [0,1] - C'$ is open, dense and has measure < 1.

But what about dynamics?

- recall the usual map on Σ_2^+ is the left shift $\sigma(.s_0 s_1 s_2 \ldots) = .s_1 s_2 \ldots$. How can we realize this on C?

- Define $f : [0, 1/3] \cup [2/3, 1] \to [0,1]$ as

$$f(x) = \begin{cases} 3x & \text{when } x \in [0, 1/3] \\ 3x - 2 & \text{when } x \in [2/3, 1] \end{cases}$$



- We now restrict f to C, the Cantor middle third.

- Claim: $f: C \to C$ and is onto

<u>Proof</u> The ~~crucial~~ crucial observation is that

$$f(I_b) = I_{\sigma(b)}$$ where for a block $b = b_0 b_1 \ldots b_n$, $\sigma(b) = b_1 \ldots b_n$

which is easily checked.

Thus if $x = \bigcap I_{b_j}$ then $f(x) = \bigcap f(I_{b_j}) = \bigcap f(I_{\sigma(b_j)})$ where we adapt the convention that if b is a single symbol, then $I_{\sigma(b)} = [0,1]$

Thus for any $y \in C$ we can find two points $x_0, x_1 \in C$ with $f(x_0) = f(x_1) = y$ i.e. if $y = \bigcap I_{b_j}$ then

$$x_0 = \bigcap I_{0b_j} \text{ and } x_1 = \bigcap I_{1b_j}$$

Now we see informally that f acts on C like σ on Σ_2^+. More formally

$$\begin{tikzcd} \Sigma_2^+ \arrow[r, "\sigma"] \arrow[d, "\varphi"'] & \Sigma_2^+ \arrow[d, "\varphi"] \\ C \arrow[r, "f|_C"'] & C \end{tikzcd}$$

commutes

<u>DEF</u>: Given $h: X \to X$ and $g: Y \to Y$ they are called topologically conjugate if there exists a homeomorphism φ with

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "\varphi"'] & X \arrow[d, "\varphi"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

commuting.

- Top conjugate dynamical systems share all aspects of their dynamics that are topological in character eg (X, h) is minimal $\Leftrightarrow$ (Y, g) is minimal

  (X, h) is transitive $\Leftrightarrow$ (Y, g) is transitive

  $O(x, h)$ is periodic $\Leftrightarrow$ $O(\varphi(x), g)$ is periodic

It is the isomorphism in the category of topological dynamical systems.

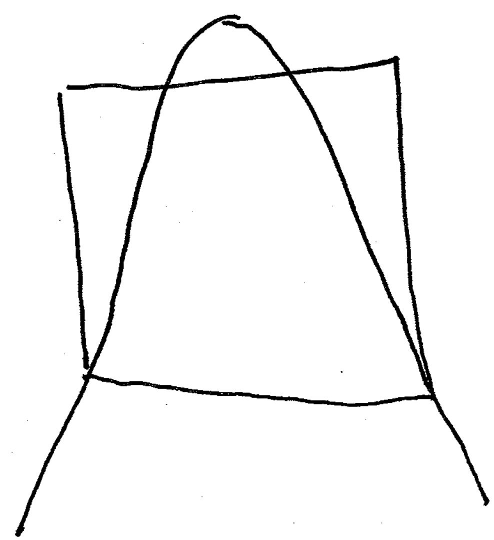
- DEF: If

$$\begin{array}{ccc} X & \xrightarrow{h} & X \\ \psi \downarrow & & \downarrow \psi \\ Y & \xrightarrow{g} & Y \end{array}$$

with ψ continuous and onto and usually not bijective then (Y, g) is <u>semiconjugate</u> to (X, h) and (Y, g) is called a <u>factor</u> of (X, h) and (X, h) and <u>extension</u> of (Y, g)

This is the morphism in the category.

Next time we will study a more realistic situation – the quadratic maps with $\mu > 4$



and show when $\mu > 2+\sqrt{5}$ that when $\Lambda = \{x : O^+(x, f_\mu) \subseteq [0,1]\}$ then $(\Lambda, f_\mu|_\Lambda)$ is topologically conjugate to (Σ_2^+, σ)