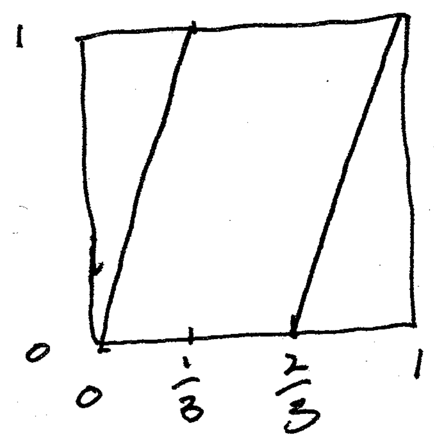
- There was an arithmetic error in the last lecture in the formula for $f: C \to C$



$$f(x) = \begin{cases} 3x & x \in [0, 1/3] \\ 3x-2 & x \in [\frac{2}{3}, 1] \end{cases}$$

- The proper formula gives another proof that f is a bijection $C \to C$ conjugate to the shift
- Recall $C \longleftrightarrow$ all base 3 expansions in $[0,1]$ with no 1

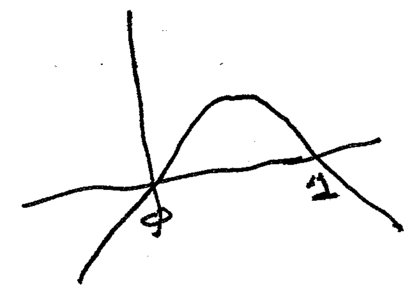
$x \in [0, 1/3] \longleftrightarrow .0 s_1 s_2 s_3$, $f(x) = 3x = .s_1 s_2 s_3$

$x \in [\frac{2}{3}, 1] \longleftrightarrow x = .2 s_1 s_2 s_3$ $f(x) = 3x - 2$

$= 2.s_1 s_2 s_3 \ldots - 2 = .s_1 s_2 s_3 \ldots$

We go back to the quadratic (logistic) family

$$f_\mu(x) = \mu x(1-x)$$



- Standing assumption: $\boxed{\mu > 1}$
- We will often just write f for f_μ with μ understood

Some elementary properties

(1) $f'(x) = \mu(1-2x)$ so $f \uparrow$ on $(-\infty, 1/2)$, $f \downarrow$ on $(\frac{1}{2}, \infty)$

(2) Fixed points: $f(p) = p$

$p = \mu p(1-p) \Rightarrow p = 0$ or $p = \dfrac{\mu - 1}{\mu}$

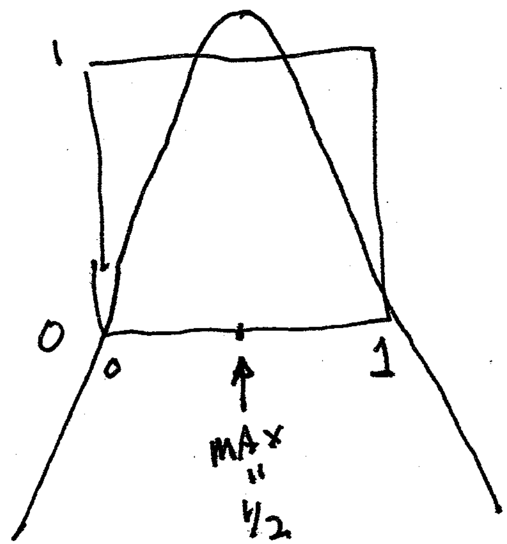
(3) $f(1) = 0$ $\quad f^2(1) = f(0) = 0$

(4) If $x \notin [0,1]$ then $f^n(x) \to -\infty$ as $n \to \infty$

<u>Proof:</u> Assume $x < 0 \Rightarrow \mu x(1-x) < x$ and so $f(x) < x$ thus $\{f^n(x)\}$ is a decreasing sequence. If it is bounded below, $\exists\, p < 0$ with $f^n(x) \to p$ and so $f^{n+1}(x) \to f(p)$ but $\{f^n(x)\}$ and $\{f^{n+1}(x)\}$ have the same tails and so $f(p) = p < 0$, contradiction to (2)

If $x > 1 \Rightarrow f(x) < 0$ using the formula for f and so $f^n(f(x)) \to -\infty$ $\blacksquare$

- The case $1 < \mu \leq 4$ requires a fair amount of machinery so we start with $\mu > 4$. This implies that $f_\mu(\frac{1}{2}) > 1$ and so $f^n(\frac{1}{2}) \to -\infty$. Graphically, the MAX is above the box $[0,1]^2$



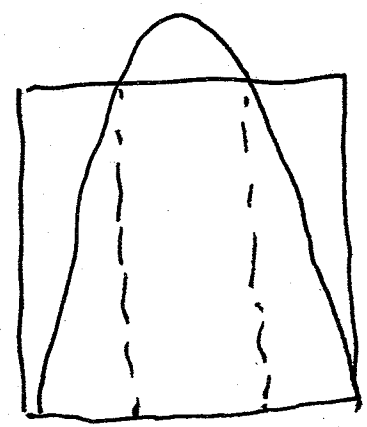
- The interesting dynamics takes place on

$$\Lambda_\mu = \{x : O(x, f_\mu) \subseteq [0,1]\}$$

i.e. $f^n(x) \in [0,1] \ \forall n \in \mathbb{N}$

5

- We describe the step by step analysing which orbits remain in $[0,1]$ and which leave after n iterates
- The crucial observation is that f has two nice branches

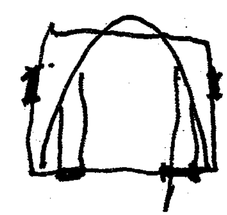


a b

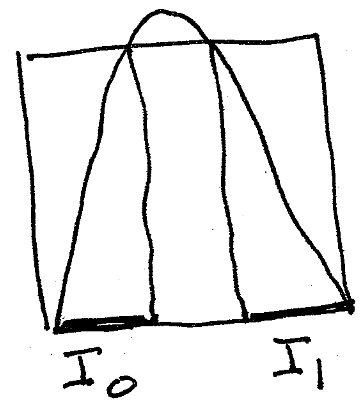
Assume $a<b$ and $f(a)=f(b)=1$

- $f_0 := f_{|[0,a]}$ maps $[0,a]$ homeomorphically onto $[0,1]$ preserves orientation
- $f_1 : f_{|[b,1]}$ " $[b,1]$ " " " reverses "
- $y \subseteq [0,1] \Rightarrow f^{-1}(y) = f_0^{-1}(y) \sqcup f_1^{-1}(y)$

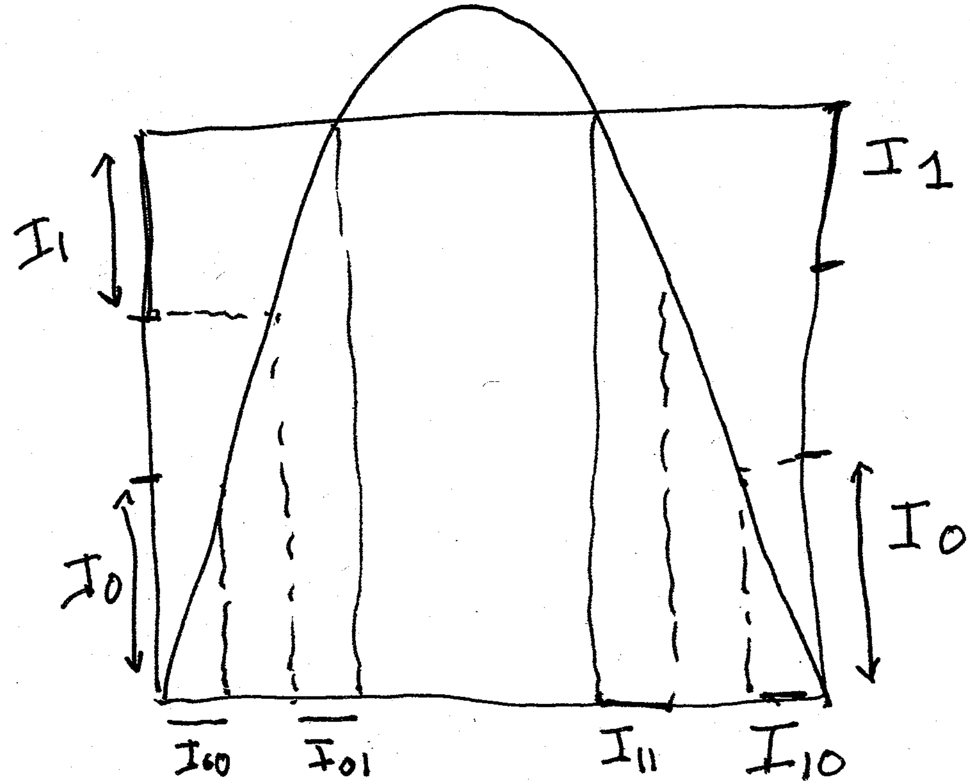
disjoint union



- Let $I_0 = [0, a]$ and $I_1 = [b, 1]$ and
  $S_1 = I_0 \amalg I_1$ and $x \in S_1 \Leftrightarrow f(x) \in [0,1]$
  or $S_1 = f^{-1}(I)$



- Let $S_2 = f^{-2}(I) = f^{-1}(I_0 \amalg I_1)$
  $= f_0^{-1}(I_0) \amalg f_0^{-1}(I_1) \amalg f_1^{-1}(I_0) \amalg f_1^{-1}(I_1)$
  $:= I_{00} \amalg I_{01} \amalg I_{10} \amalg I_{11}$



: NOTE : • $I_{11} < I_{10}$

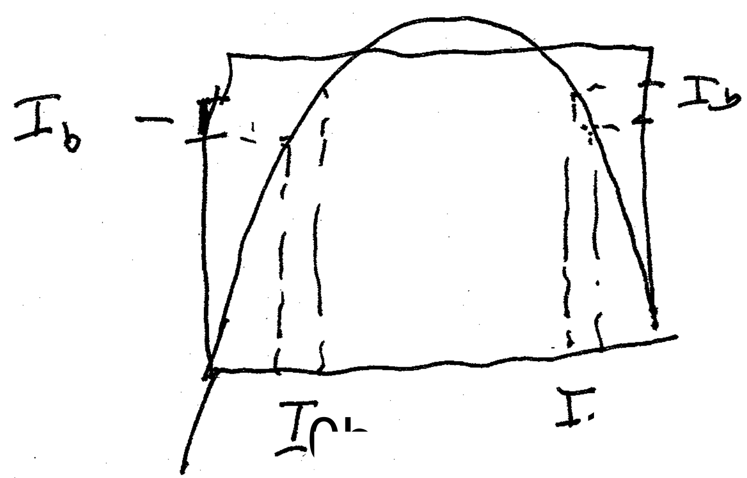
• $x \in I_{\alpha\beta} \Leftrightarrow x \in I_\alpha \quad f(x) \in I_\beta$

• $I_{\alpha\beta} = \underset{\alpha}{f^{-1}}\,\underset{\beta}{f^{-1}}(I)$

Proceeding, for a length n block

$b = s_0 s_1 \ldots s_{n-1}$

Let $I_b = f_{s_0}^{-1} \circ \cdots \circ f_{s_{n-2}}^{-1} \circ f_{s_{n-1}}^{-1}(I)$



$S_n = \coprod \{ I_b : b \in \{0,1\}^n \} = f^{-n}(I)$

$x \in I_b \Leftrightarrow x \in I_{s_0}, f(x) \in I_{s_1}, \ldots f^{n-1}(x) \in I_{s_{n-1}}$

- The point of the construction is that

$$\Lambda = \bigcap_{n=0}^{\infty} S_n = \{x : O(x,f) \subseteq [0,1]\}.$$

- The process throws out "middle thirds" but is Λ a Cantor set
- It is compact since each S_n is a finite union of closed intervals and so is compact and $S_{n+1} \subseteq S_n \Rightarrow \bigcap S_n$ is compact.
- Each component of Λ is coded by $\underline{s} \in \{0,1\}^{\mathbb{N}} = \Sigma_2^+$ as

$$\boxed{s(0{:}n) = s\_0\ s\_1 \ldots s\_n} \quad \bigcap_{n=0}^{\infty} I_{s(0:n)}$$

but are these points

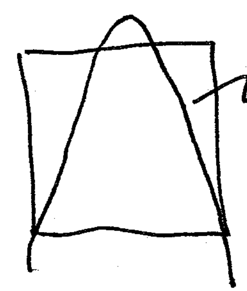
i.e. does $|I_{s(0:n)}| \to 0$ as $n \to \infty$?

- This is true for $\mu > 4$ but requires complex variable for $4 < \mu \leq 2+\sqrt{5}$

$\mu > 2+\sqrt{5} \Rightarrow$ exercise

- Assume ~~now~~ that

$|f'(x)| > 1$ when $x \in I_0 \cup I_1 = [0,a] \cup [b,1]$



- We need a few Calculus Lemmas

(1) Assume ~~$|f_k'(x)| < \lambda < 1$~~ ~~$g_k : I \to I$ and~~

$g_k : I \to I$ with $|g_k'| < \lambda < 1$ $\Rightarrow$ If

$g = g_1 \circ g_2 \circ \cdots g_n \Rightarrow |g'| < \lambda^k$

<u>Proof</u> chain Rule $g'(x) = g_1'(g_2 \circ \cdots g_n(x)) \cdots \quad g_n'(x)$

TAKE ABSOLUTE VALUE

(2) Assume $g : [a,b] \to [c,d]$ is bijective and bi differentiable with $|g'(x)| > \lambda > 1$ for all $x \in [a,b]$.

If $J \subseteq [a,b]$ is a subinterval $\Rightarrow |g(J)| > \lambda |J|$

and if $K \subseteq [c,d]$, $|g^{-1}(K)| < 1/\lambda\, |K|$

<u>Proof</u> If $J = [\alpha, \beta]$, by the Mean Value Theorem $\exists\, x_0 \in J$ with $\dfrac{g(\beta) - g(\alpha)}{\beta - \alpha} = g'(x_0)$. Thus

Since g is bijective This implies $\dfrac{|g(J)|}{|J|} = \dfrac{|g(\beta) - g(\alpha)|}{|\beta - \alpha|} =$

$|g'(x_0)| > \lambda$. For the second part, since g is bi-differentiable

$g\, g^{-1}(x) = x$ implies $g'(g^{-1}(x)) \cdot (g^{-1})'(x) = 1$, so

so $|(g^{-1})'(x)| = \left|\dfrac{1}{g'(g^{-1}(x))}\right| < 1/\lambda$ Then same MVT proof.

<u>Lemma</u>: If $\underline{s} \in \Sigma_2^1$ then

$$|I_{s_0 s_1 \ldots s_{n-1}}| \to 0 \quad \text{as } n \to \infty$$

<u>Proof</u> $I_{s_0 \ldots s_{n-1}} = f_{s_0}^{-1} \circ \ldots f_{s_{n-1}}^{-1}(I)$

each f_i^{-1} is bijective, bi-differentiable

and $|(f_i^{-1})'| < \frac{1}{\lambda} < 1 \Rightarrow$ by the facts above

$|I_{s_0 \ldots s_{n-1}}| < \frac{1}{\lambda}^n \to 0$ as $n \to \infty$.

<u>CORR</u> Λ is a Cantor space (perfect, compact, completely disconnected.)

Proof is similar to Cantor middle third

- What about dynamics?