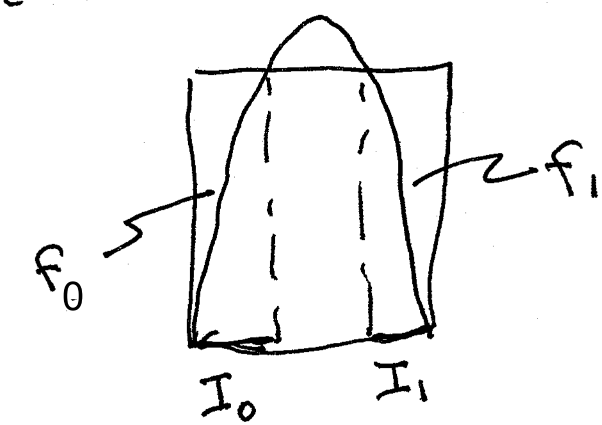
- There WAS an indexing mistake in the last video - it is corrected in the hard copy

recall



If $b = s_0 \cdots s_{n-1}$, $I_b = f_{s_0}^{-1} f_{s_1}^{-1} \cdots f_{s_{n-1}}^{-1}(I)$

and $x \in I_b \iff x \in I_{s_0}, f(x) \in I_{s_1}, \ldots, f^{n-1}(x) \in I_{s_{n-1}}$

Addresses and Itineraries

- We introdue now a frequently used tool in Dynamics which takes a Dynamical System $f: X \to X$ and associates it with a symbolic system
- Alternatively, it takes an orbit $o(x,f)$ and produces a signal
- We will continue to just work with forward orbits and non-injective f
- For $n \in \mathbb{N}$, $\Sigma_n^+ = \{0,1,..,n-1\}^{\mathbb{N}}$

$$= \{.s_0 s_1 s_2 .. : s_i \in \{0,1,..,n-1\}\}$$

This is the <u>full, one-sided n-shift</u>

- The left shift map is the same $\sigma: \Sigma_n^+ \to \Sigma_n^+$

via $\sigma(.s_0 s_1 \ldots) = .s_1 s_2 \ldots$

- It is continuous and n-to-one.

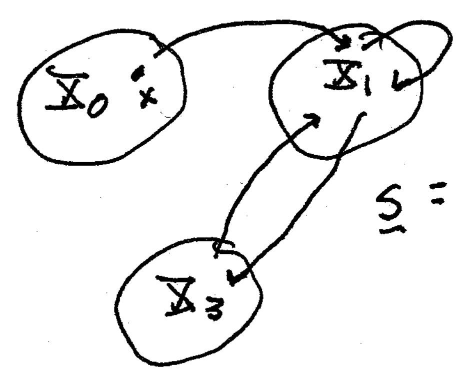
- Now Assume $f: X \to X$ is onto and we have a partition of X into disjoint sets

$$X = X_0 \sqcup X_1 \sqcup \ldots \sqcup X_{n-1}$$

- The address function is $A(x) = j$ if $x \in X_j$.
- The itinerary map is $i: X \to \Sigma_n^+$ via

$(i(x))_j = A(f^j(x))$ so $i(x) = \underline{s} = .s_0 s_1 \ldots$

means $f^j(x) \in X_{s_j}$

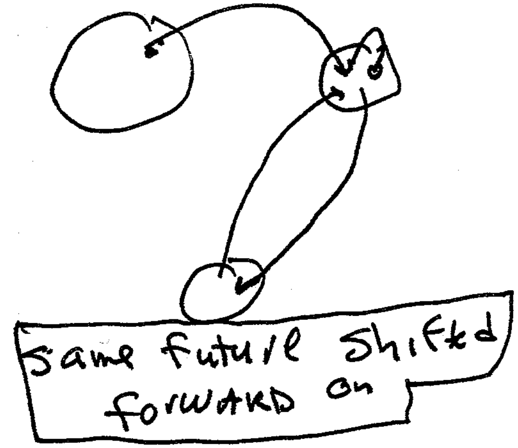


$\underline{s} = 01131\ldots$

- The fundamental property is

diagram for semi-conjugacy

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ i\downarrow & & \downarrow i \\ \Sigma_n^+ & \xrightarrow{\sigma} & \Sigma_n^+ \end{array}$$



commutes or $i \circ f = \sigma \circ i$

<u>Proof</u>: Say $i f(x) = \underline{s}$. This happens exactly when $f^j(f(x)) \in X_{s_j}$ or $f^{j+1}(x) \in X_{s_j}$. This happens

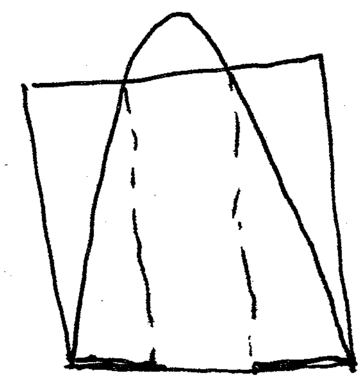
$\iff (i(x))_{j+1} = s_j \iff \sigma(i(x))_j = s_j$

$\iff \sigma i(x) = \underline{s}$

- BASIC QUESTIONS: Is i continuous, injective, what is its image? These depend on X and f of course.

- Often one does not have a nice decomposition of Σ into disjoint sets (Σ is connected) and one has to deal with overlap of the addresses.

- We return to the quadratic map with $\mu > 2+\sqrt{5}$.



- $\Lambda = \{x : O(x,f) \subseteq [0,1]\}$

- our address system is

$$\Sigma_0 = I_0 \cap \Lambda$$
$$\Sigma_1 = I_1 \cap \Lambda$$

Theorem When $\mu > 2+\sqrt{5}$, $i : \Lambda \to \Sigma_2^+$ is a homeomorphism (so is onto) and thus provides a topological conjugacy

$$\begin{array}{ccc} \Lambda & \xrightarrow{f|_\Lambda} & \Lambda \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle i} \\ \Sigma_2^+ & \xrightarrow{\sigma} & \Sigma_2^+ \end{array}$$

Application: We know (Σ_2^+, σ) is transitive (has a dense forward orbit) so $(\Lambda, f|_\Lambda)$ is transitive.

Proof: Recall if $b = s_0 \ldots s_{n-1}$ then

$$x \in I_b \iff x \in I_{s_0},\ f(x) \in I_{s_1}, \ldots, f^{n-1}(x) \in I_{s_{n-1}}$$

which is to say $(i(x))_j = s_j$ for $j = 0, \ldots, n-1$

so the initial block of the itinerary of x is b

The construction of the I_b ensures that

$\Lambda = \{ \bigcap_{n=0}^{\infty} I_{s(0:n)} : \underline{s} \in \Sigma_2^+ \}$. Since

$I_{b0} \amalg I_{b1} \subseteq I_b$ each $\bigcap_{n=0}^{\infty} I_{s(0:n)} \neq \emptyset$

and we showed they were points. Thus we have a

well-defined function $I : \Sigma_2^+ \to \Lambda$ defined by

$I(\underline{s}) = \bigcap_{n=0}^{\infty} I_{s(0:n)}$ which is bijective.

As with the Cantor set proof, $\underline{s}^{(n)} \to \underline{s}$

implies $I(\underline{s}^{(n)}) \to I(\underline{s})$ so I is continuous.

Using the initial remark, $i\, I(\underline{s}) = \underline{s}$ and $I\, i(x) = x$

so $I = i^{-1}$. Finally, i is also continuous

via a direct argument or the next fact.

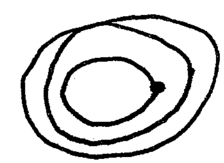
The diagram commutes since i is an itinerary. $\blacksquare$

This is a useful topological fact

- If X, Y are metric, X is compact, $g: X \to Y$ is bijective and continuous $\Rightarrow g^{-1}$ is continuous so g is a homeomorphism.

$\boxed{\text{Another example - The angle doubling map}}$

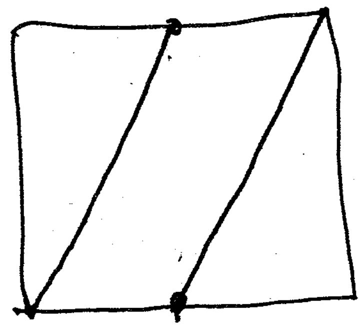
- 1st definition: $S^1 = \{z \in \mathbb{C} : |z| = 1\}$
  $d: S^1 \to S^1$, $d(z) = z^2$



- 2nd definition: Parameterize S^1 as $e^{i\theta}$
  Then $d: e^{i\theta} \mapsto e^{2i\theta}$

- Treat $S^1 = [0,1]/0 \sim 1$ and $d(x) = 2x \bmod 1$

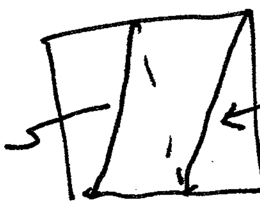
- The last definition allows us to draw a graph



$d(0) = 0$ but $0 \sim 1$
$d(1) = 1$
$d(1/2) = 1 \equiv 0$

- We want to code d using addresses and itineraries
- The natural choice is

$I_0 = [0, 1/2]$
$I_1 = [1/2, 1]$

and d has two ~~~~ expanding branches d_0  d_1

- Issues: • what about overlap at $1/2$ and $0 \equiv 1$
  - We can't have a continuous map
    $i : [0,1]/\sim \;\to \Sigma_2^+$ since S^1 is connected and Σ_2^+ is completely disconnected

- There are several ways to deal with this, one way is to start with the "Good set"

$$G = \{ x \in [0,1] : d^n(x) \neq 1/2 \text{ or } 0 \text{ or } 1 \}$$

Now if $d^n(x) = 1/2 \Rightarrow d^{n+1}(x) = [0]$ the class of zero

so to be simpler

$$G = \{ x \in [0,1] : d^n(x) \neq [0] \ \forall n \in \mathbb{N} \}$$

- We can figure out exactly what G is

$$d^{-1}([0]) = 1/2, \quad d^{-1}(1/2) = 1/4, 3/4, \ldots$$

$$\text{So } G = S^1 - \left\{ \frac{a}{2^n} : 0 \leq a \leq 2^n, a \in \mathbb{N} \right\}$$

a countable set removed

- Now define $i : G \to \Sigma_2$ via

  $(i(x))_j = A(d^j(x))$ where $A(x) = 0 \quad x \in I_0 \cap G$
  $\qquad\qquad\qquad\qquad\qquad\quad\;\, = 1 \quad x \in I_1 \cap G$

- Since $d'(x) = 2$ on $(0, 1/2) \cup (1/2, 1)$
  [or all of S^1 with the other def]

  Things will work like the quadratic map

- **Lemma**: $i : G \to \Sigma_2$ is injective and continuous.

- What is $i(G)$?

-

- What do we miss? Answ: $.0000^\infty$ and $.111^\infty$

and anything that ends up in these

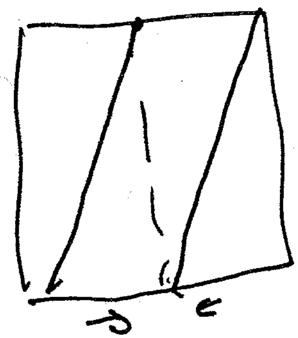
$$\text{Image}(i) = \Sigma_2^+ - \{\underline{s} : \underline{s} = b0^\infty \text{ or } \underline{s} = b1^\infty \text{ for some block } b\}$$

- One way to proceed let $I : \text{image}(i) \to G$ be i^{-1}

- Show
  - (a) I is uniformly continuous
  - (b) image(i) is dense in Σ_2^+ (since it is Σ_2^+ minus a countable set)
  - (c) Thus I extends to a continuous

$$\begin{tikzcd} \Sigma_2^+ \arrow[r, "\sigma"] \arrow[d, "I"'] & \Sigma_2^+ \arrow[d, "I"] \\ S^1 \arrow[r, "d"] & S^1 \end{tikzcd}$$

- It will be a semiconjugacy since I is 1-1 except on a countable set where it is 2-1.



$I^{-1}(1/2) = \{.0111^{\infty}, 1000^{\infty}\}$

limit from left limit from right

This should look like a familiar ambiguity and leads to the second method that doesn't generalize but works well in this case.

- use base 2 expansions, explicitly

$$I(\underline{s}) = \sum_{j=0}^{\infty} \frac{s_j}{2^{j+1}}$$

Notice $I(.01^{\infty}) = \sum_{j=1}^{\infty} \frac{1}{2^{j+1}} = 1/2 = I(10^{\infty})$

- more next time.