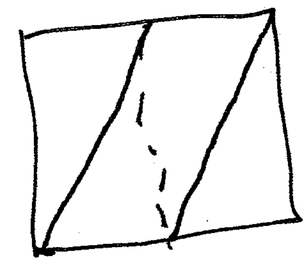
DS8

- Recall $d : S^1 \to S^1$ via $z \mapsto z^2$ on $\{|z|=1\} \subseteq \mathbb{C}$

or $x \mapsto 2x \bmod 1$ on $[0,1]/\sim$



- Define $I : \Sigma_2^+ \to [0,1]$ via

$$I(\underline{s}) = \sum_{j=0}^{\infty} \frac{s_j}{2^{j+1}}$$ so $I(\underline{s})$ has base 2 expansion $.s_0 s_1 \ldots$

- **Theorem**: I is continuous, onto and provides a semi conjugacy

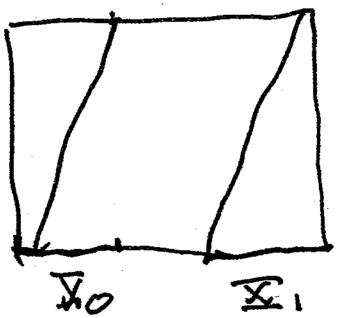
$$\begin{tikzcd} \Sigma_2^+ \arrow[r, "\sigma"] \arrow[d, "I"'] & \Sigma_2^+ \arrow[d, "I"] \\ {[0,1]/\sim} \arrow[r, "d"] & {[0,1]/\sim} \end{tikzcd}$$

<u>Proof</u> continuous and onto are straight forward for the commutivity of diagram

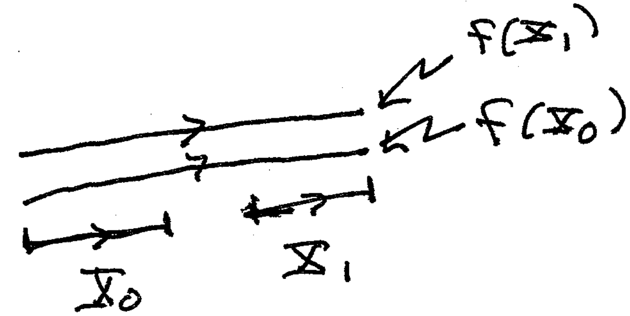
$$
\begin{aligned}
d\,I(\underline{s}) &= 2 \sum_{j=0}^{\infty} \frac{s_j}{2^{j+1}} \mod 1 \\
&= \sum_{j=0}^{\infty} \frac{s_j}{2^j} \mod 1 \\
&= s_0 + \sum_{j=0}^{\infty} \frac{s_{j+1}}{2^{j+1}} \mod 1 \\
&= \sum_{j=0}^{\infty} \frac{s_{j+1}}{2^{j+1}} \mod 1 = I\,\sigma(\underline{s}) \quad \blacksquare
\end{aligned}
$$

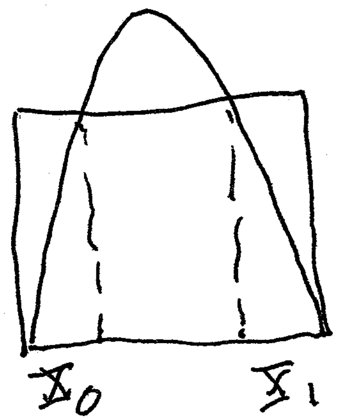
- The pairs $.b01^{\infty}$ and $.b10^{\infty}$ are the only places I is 2-1
- When we have measures we will see they are conjugate almost everywhere - same <u>statistics</u>.

examples so for ~~co~~ (semi) conjugate to (Σ_2^+, σ)

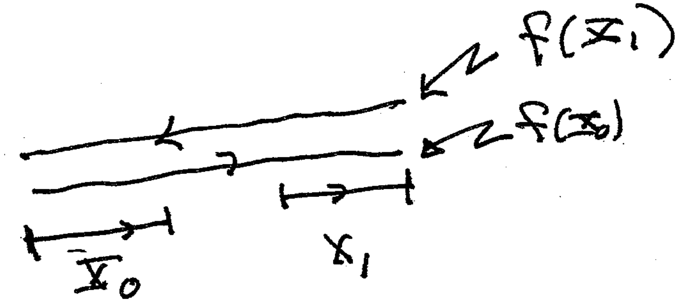


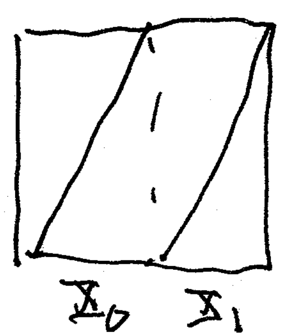
f acting on C



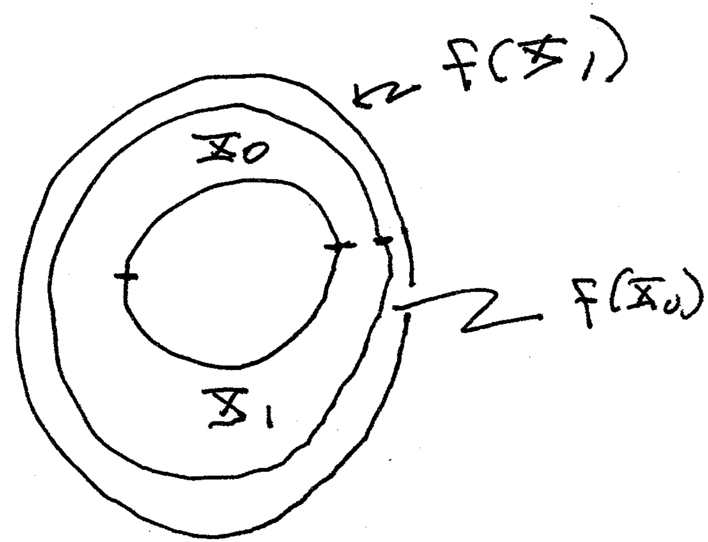


f_μ acting on Λ





d acting on S^1



In all cases:

Ξ_0 covers itself and Ξ_1

Ξ_1 covers itself and Ξ_0

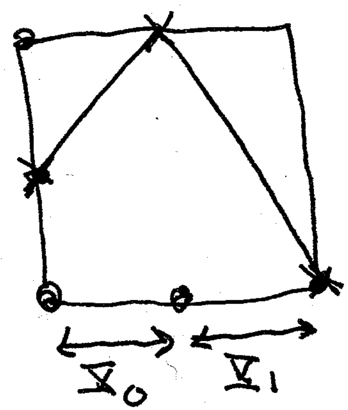
$$\begin{tikzcd} \Xi_0 \arrow[loop above] \arrow[r, bend left] & \Xi_1 \arrow[loop above] \arrow[l, bend left] \end{tikzcd}$$

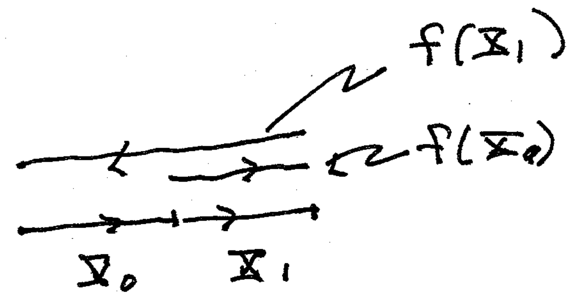
Think of Ξ_i as states (Addresses)
and arrows are allowable transitions.

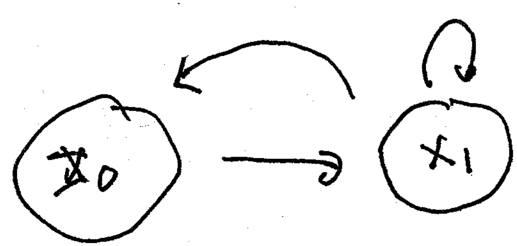
- Consider a new situation we have a periodic orbit

of period 3

- We construct the simplest map which has this orbit







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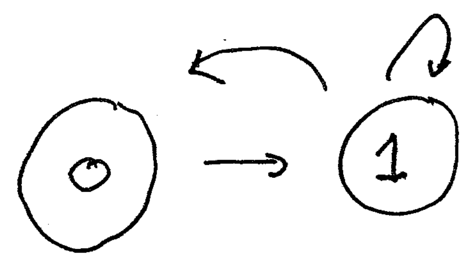
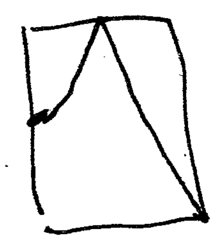
- It is usual to describe this situation by a transition matrix

$$\begin{array}{c|cc} & \Phi_0 & \Phi_1 \\ \hline \Phi_0 & 0 & 1 \\ \Phi_1 & 1 & 1 \end{array} \quad \text{where} \quad M_{ij} = 1 \text{ if } \Phi_i \to \Phi_j$$
$$= 0 \text{ if } \Phi_i \not\to \Phi_j$$

[some books/fields use the transpose.]





- What kind of itineraries do we expect - 00 not allowed
- Let $\Omega = \{ \underline{s} : s_i = 0 \; s_{i+1} = 0$ never occurs $\}$

Ω is closed, σ-invariant called a <u>Subshift of finite type</u> ssft <u>or Topological Markov Chain</u>

- Recall $\Sigma_n^+ = \{.s_0 s_1 \ldots : s_i \in \{0, \ldots, n-1\}\}$
  $= \{0, \ldots, n-1\}^{\mathbb{N}}$

- $X \subseteq \Sigma_n^+$ with $\sigma(X) = X$ is called a subshift

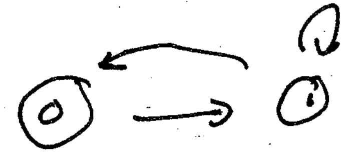
- **DEF:** Given a $(n \times n)$ matrix A with $A_{i,j} \in \{0,1\}$ $\forall i,j$ the corresponding subshift of finite type (ssft) is $\Sigma_A^+ = \{\underline{s} \in \Sigma_n^+ : A_{s_i s_{i+1}} = 1 \ \forall i\}$

- $$\begin{array}{cc} & j \\ & \downarrow \\ i \rightarrow & \begin{bmatrix} 1 \\ \\ \end{bmatrix} \end{array}$$

  if $A_{ij} = 1$ ij can occur next to each other
  - called allowable transition

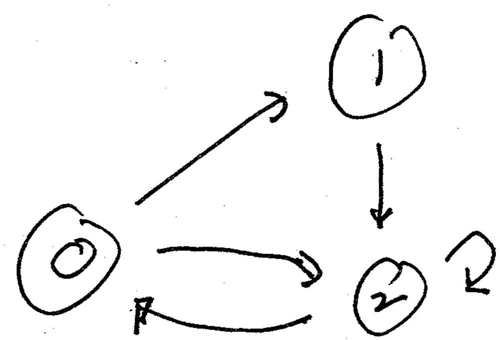
  $A_{ij} = 0$, ij can not occur,
  - called forbidden transition

- In our example above



$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$$

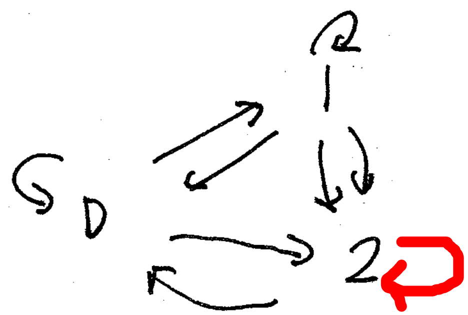
- In general, a ssft may also be specified by a <u>directed</u> graph on n vertices. The corresponding matrix is the transition or adjacency matrix



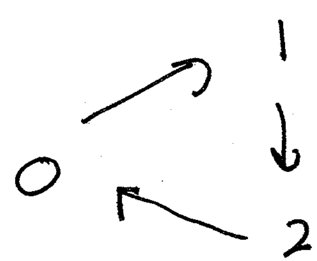
$$A = \begin{bmatrix} 0 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

There is a directed edge for vertex i to vertex j $\iff A_{ij} = 1$. Then allowable sequences are paths through the graph.

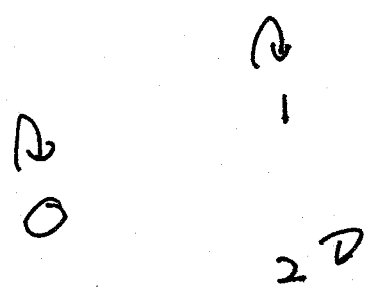
Some examples



$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ $\Sigma_A^+ = \Sigma_3^+$ the full three shift

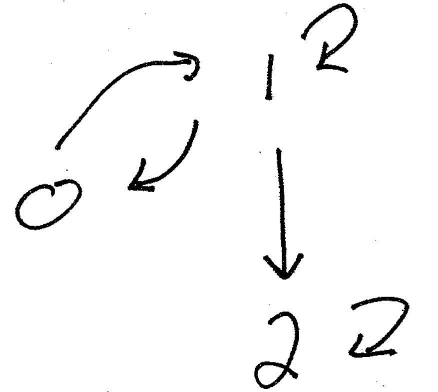


$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ a permutation matrix $\Sigma_A^+ =$ a single period 3 pt $.012\,012\,(012)^\infty$ and its orbit



$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ the identity

$\Sigma_A^+ = 3$ fixed pts $0^\infty, 1^\infty, 2^\infty$



$$A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

2 is an absorbing state.

FACT: For any A, Σ_A is a compact, σ invariant subset of Σ_n^+

NOTE: One may also define 2-sided shift.

Question: How do the properties of the matrix or graph determine the dynamics of (Σ_A, σ). When is it transitive? How many periodic points are there? etc.