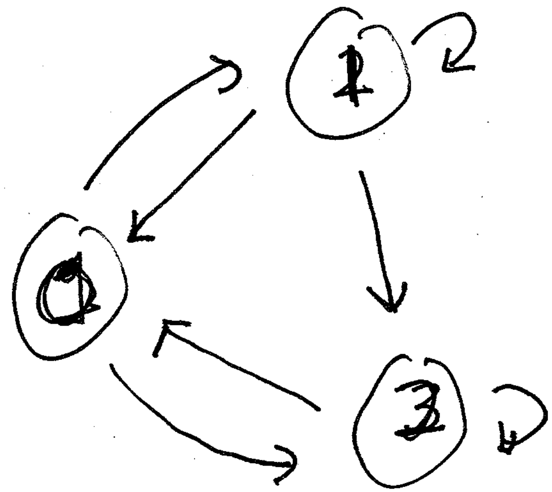
SSFT continued

- A is an $n \times n$ $\{0,1\}$-matrix indexed 0, …, n-1
- $\Sigma_A^+ = \{ \underline{s} \in \Sigma_n^+ : A_{s_i s_{i+1}} = 1 \text{ for all } i \}$
- If $A_{ij} = 1$ $\quad ij$ is allowable
   $A_{ij} = 0$ $\quad ij$ is forbidden.
- Σ_A may also be specified by a directed graph on n numbered vertices

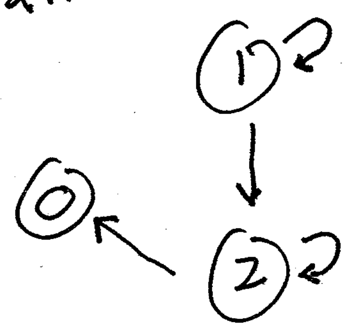
  $A_{ij} = 1 \iff$ directed edge for i to j



$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

- An allowable word (or block) $b = s_0 s_1 \ldots s_{n-1}$ is such that each $s_i s_j$ is allowable

- An allowble word $\longleftrightarrow$ path is the graph.

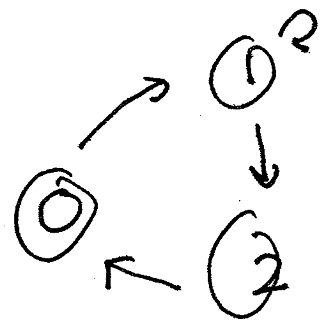


$1\,1220$ is length 5 allowable word.

- What are periodic points in Σ_A^+?

- $\underline{s} = (s_0 \ldots s_{n-1})(s_0 \ldots s_{n-1})(s_0 \ldots s_{n-1}) \Longleftrightarrow \sigma^n(\underline{s}) = \underline{s}$

- So $s_0 \ldots s_{n-1}$ is allowable $\underline{\underline{\text{and}}}$ $s_{n-1} s_0$ is also

- 

  So a periodic point is a path in the graph that ends where it starts

  $(0112)(0112)\ldots(0112)^{\infty}$

- So periodic points $\Longleftrightarrow$ loops in the graph

- How do we count allowable words (the language of the shift) and periodic points. MATRIX MULTIPLICATION

- <u>Lemma</u>: The number of allowable words (in Σ_A^+) of length n (= length n paths) starting from i and ending with j is $(A^n)_{ij}$ i.e. (ij)-entry of A^n.

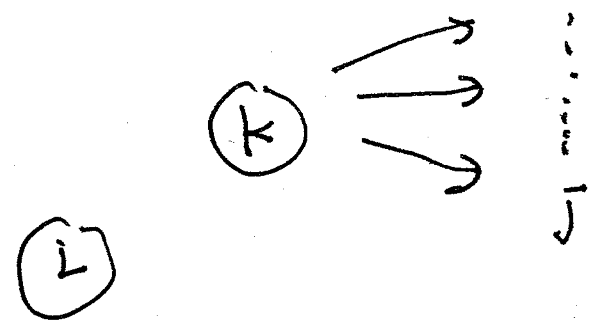
<u>Proof</u> Induction on n. The case $n=0$ is by definition of A. Now write $A^n = A(A^{n-1})$

Then $$(A^n)_{ij} = (A_{i1}\ A_{i2} \cdots A_{in}) \begin{pmatrix} (A^{n-1})_{1j} \\ \vdots \\ (A^{n-1})_{nj} \end{pmatrix}$$

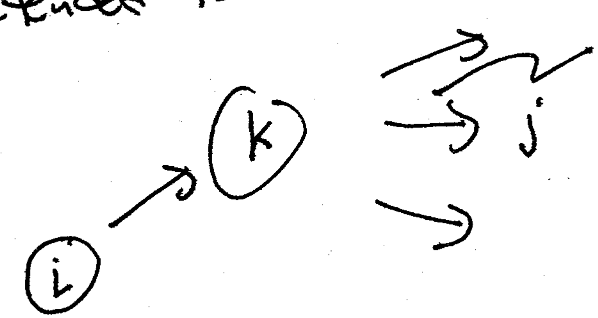
$$= \sum_{k=0}^{n-1} A_{ik} (A^{n-1})_{kj}.$$

What does each term mean?

If $A_{ik} = 0$ then ik is not allowable, and so there is no path starting ik and continuing



If $A_{ik} = 1$, ik is allowable and each allowable path $k \to j$ of length $n-1$ (as measured by $(A^{n-1})_{kj}$) can be extended to start at i



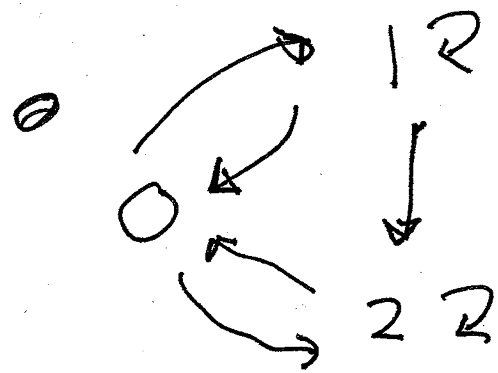
$\# = A^{n-1}_{kj}$

So summing yields all length n paths $i \to j$. ∎

- A point $x \in X$ has period n under f if n is the <u>least</u> positive integer so that $f^n(x) = x$.
- If $f^n(x) = x \Rightarrow \forall k,\ f^{nk}(x) = x$
- $\operatorname{Fix}(f^n)$ = all fixed points of f^n includes all points with period k with $k \mid n$.
- Some books say x has period n if $f^n(x) = x$, but we always mean least period
- BACK TO Σ_A^+, a loop of length n traversed <u>once</u> yields a periodic point of period n
- Any loop (perhaps with repeated trips) of length n yields a point in $\operatorname{Fix}(\sigma^n)$

- So any allowable block $b = s_0 s_1 \dots s_{n-1}$ with $s_{n-1} s_0$ allowable yields a sequence $b^\infty \in \mathrm{Fix}(\sigma^n, \Sigma_A)$

- eg



$b = 012$ yields a period 3 point

$b = 012012$ yields $\underline{s} \in \mathrm{Fix}(\sigma^6, \Sigma_A)$

$b = 1120$ concatenates 1^∞ $(120)^\infty$ (more later)

- **CORR:** $\#(\mathrm{Fix}(\sigma^k, \Sigma_A)) = \sum_{j=0}^{n-1} A^k_{jj} = \mathrm{Trace}(A^k)$.

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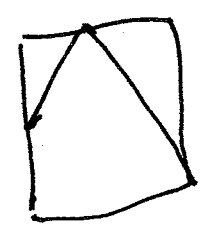
- Some linear Algebra $tr(A) = \lambda_1 + \dots + \lambda_n$ where $\{\lambda_i\}$ are the eigenvalues of A
- So $trace(A^k) = \lambda_1^k + \dots + \lambda_n^k$
- Assume $|\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| \Rightarrow \rho(A)$ = spectral radius of $A = |\lambda_1|$
- In Σ_A^+, $\# Fix(\sigma^k) \sim (\rho(A))^k$
- $\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[loop right] \end{tikzcd}$ from  $A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$

$$\lambda = \frac{1 \pm \sqrt{1^2 - 4}}{2} = \frac{1 \pm \sqrt{5}}{2}, \quad \rho(A) = \frac{1 + \sqrt{5}}{2}$$

so $Fix(\sigma^k, \Sigma_A^+) \sim \left(\frac{1+\sqrt{5}}{2}\right)^k$

- $$\begin{tikzcd} 0 \arrow[r, bend left] & 1 \arrow[l, bend left] \end{tikzcd} \qquad A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad \lambda = \frac{0 \pm \sqrt{-4}}{2} = \pm i$$

  $\rho(A) = 1$ no growth $\left(\Sigma_A = \{(01)^\infty, (10)^\infty\} \right)$

- $$\begin{tikzcd} 0 \arrow[loop above] \arrow[r, bend left] & 1 \arrow[l, bend left] \arrow[loop above] \end{tikzcd} \qquad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$ full two shift, $\lambda = 2, 0$,

  # words of length $n = 2^n + 0^n = 2^n$

$\boxed{\text{We make a connection to combinatorics and number theory.}}$

- Assume $\mathrm{Fix}(f^n)$ is finite and let

$$N_n = \#(\mathrm{Fix}(f^n))$$

- The generating function of N_n ~~is~~ (or f) is

$$g_f(z) = \sum_{n=1}^{\infty} N_n z^n$$

- The zeta function of N_n (or f) is

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{1}{n} N_n z^n\right)$$

- $\zeta_f(z) = \exp(z\, g'(z))$

Theorem: The zeta function for a ssft (Σ_A, σ) is rational and in fact

$$\zeta(z) = \frac{1}{\det(I - zA)}$$

Rk: ζ has a pole where $\det(I - zA)$ is zero

or $I - zA$ is singular or $A - \frac{1}{z}I$ is singular

or $\frac{1}{z}$ is an eigenvalue of A ~~or z~~

Eg: $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\zeta(z) = \cancel{\frac{1}{z(z-2)}}\ \frac{1}{1-2z} = \exp\left(\sum \frac{2^n}{n} z^n\right)$

pole $z = 1/2$, $\lambda = 2$

$$g(z) = \sum_{n=1}^{\infty} 2^n z^n$$

For proof we need some linear algebra

(1) A is $n \times n \Rightarrow e^A = I + A + \frac{A^2}{2} + \dots + \frac{A^n}{n!} + \dots$

so $e^{At}\vec{x}_0$ solves matrix ODE $\frac{d\vec{x}}{dt} = A\vec{x}$, $\vec{x}(0) = x_0$

(2) $\det(e^B) = e^{tr(B)}$ (nontrivial)

(3) Recall $N_n = trace(A^n)$ so

$$\zeta(z) = \exp\left(\sum_{n=1}^{\infty} \frac{tr(A^n) z^n}{n}\right) = \exp\left(tr\left(\sum_{n=1}^{\infty} \frac{A^n z^n}{n}\right)\right)$$

$$= \exp(tr(-\log(I - zA))) = \exp tr \log((I - zA)^{-1})$$

$$\overset{\uparrow}{=} \det e^{\log(I-zA)^{-1}} = \det((I - zA)^{-1})$$

$$= 1/\det(I - zA)$$

$\boxed{\text{matrix } \log = \exp^{-1}}$