Ergodic Theory - Informally

- Field has its origin in classical Hamiltonian mechanics.
- When there is no friction or energy dissapation the solns to the differential equations (the solution flow) preserve a measure, the Liouville measure.
- Especially in statistical physics one wants to understand the probability of certain dynamics happening.
- A central question was the equality of time and space averages (formalized soon)

- Boltzman (1844-1906) coined the word "ergodic" from two Greek words
  ergon = work and odos (path)

- The original "ergodic hypothesis" was that each orbit of the Hamiltonian flow equaled the whole energy surface.

- We now know this is false by simple topology.

- Birkhoff and von Neumann (1931) formalized the notion of ergodicity and each proved a fundamental theorem about it.

Categories of ergodic Theory - still informal

- $(X, \mathcal{B}, \mu)$ is an abstract measure space and $f: X \to X$ is a measure preserving transformation
- X is a metric space, μ is a Borel measure, $f: X \to X$ is continuous and measure preserving.
- X is a manifold, μ is a smooth measure, $f: X \to X$ is smooth and measure preserving.

BASIC Questions

- When are two systems isomorphic, morphic?
- How do the measure theoretic and topological/smooth structure interact and influence the dynamics
- If $\alpha : X \to \mathbb{R}$ is an observable when is

$$\lim_{n \to \infty} \frac{1}{n} \underset{\underset{\text{time average}}{\uparrow}}{\sum_{j=0}^{n-1}} \alpha(f^j(x)) = \underset{\underset{\text{space average}}{\uparrow}}{\int_X \alpha \, d\mu}$$

almost everywhere in $x, d\mu$?

- Applications to typical behaviours - dynamics, number theory, differential geometry, math Bio, ---

A quick trip through measure theory

- X is a set. $\mathcal{B}$ is a collection of subsets of X. It is called a σ-algebra if

  (a) $X \in \mathcal{B}$

  closed under complements (b) $B \in \mathcal{B}$ then $X - B = B^c \in \mathcal{B}$

  closed under countable unions (c) If $B_n \in \mathcal{B}$ $n = 1, 2, \ldots$

  $$\Rightarrow \bigcup_{n=1}^{\infty} B_n \in \mathcal{B}$$

  (This is called a σ-ring in Rudin, but σ-algebra is the more common term)

- HW: If $B_n \in \mathcal{B}$, $n = 1, 2, \ldots \Rightarrow \bigcap_{n=1}^{\infty} B_n \in \mathcal{B}$.

- The pair $(X, \mathcal{B})$ is called a measurable space and elements B are called measurable sets

- The <u>Borel σ- algebra</u> on a topological space X is the smallest σ-algebra containing all the open sets in X.

- A finite measure is a function

  $\mu : \mathcal{B} \to [0,\infty)$ such that

  (a) $\mu(\phi) = 0$ (ϕ = emptyset)

  (b) If $B_n \in \mathcal{B}$ are pairwise disjoint ($B_i \cap B_j = \phi$ if $i \neq j$) then

  $$\mu\left(\bigcup_{n=1}^{\infty} B_n\right) = \sum_{n=1}^{\infty} \mu(B_n)$$

- Example: Lebesgue measure on $[0,1]$, $\mu([a,b]) = b-a$

- Sometimes it is useful to allow infinite measures (Lebesque on $\mathbb{R}$), signed measures, complex valued measures, ...

- μ is called a probability measure (or just a probability) if $\mu(\mathbb{X})=1$

- A triple $(\mathbb{X}, \mathcal{B}, \mu)$ is called a measure space. If μ is probability, it is called a probalty Space

- We shall usually work with probability spaces. note that if μ is a finite measur $\mu/\mu(\mathbb{X})$ is probability

- There is more to say about measure spaces; completion, generating measures on σ-alg from functions on algebras, ... products, . ., integration

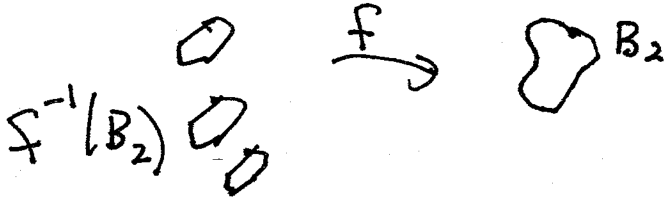
Measure preserving transformations

Suppose $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ are probability spaces

and $f: X_1 \to X_2$

(a) f is measurable if $f^{-1}(B_2) \in \mathcal{B}_1$ $\forall B_2 \in \mathcal{B}_2$

(b) f is measure preserving if it is measurable and $\forall B_2 \in \mathcal{B}_2$, $\mu_1(f^{-1}(B_2)) = \mu_2(B_2)$



(c) f is an invertible measure preserving transformation, if it is bijective, measure preserving and f^{-1} is also measure preserving.

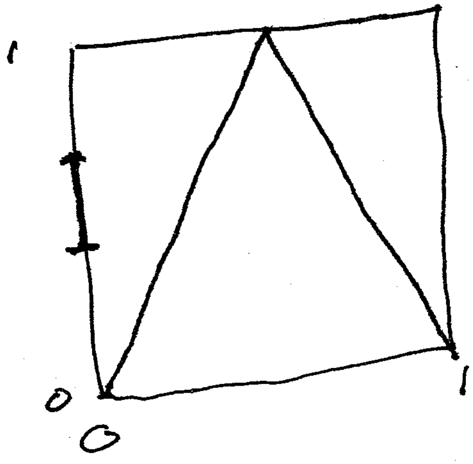
Why the inverse?

- for example, if f is continuous with respect to two topologies and $\mathcal{B}_i$ are Borel sigma algebras f^{-1} of an open set is open but f of an open set is perhaps not open or even Borel measurable

- f is perhaps not invertible

- In formal example. full tent map

$$f(x)=2x \quad 0 \le x \le 1/2$$
$$=-2x-2 \quad 1/2 \le x \le 1$$



$f^{-1}([a,b])$ is two intervals each with length $|b-a|/2$ since $|f'|=2$ everywhere

So f preserves Lebesgue measure.

SELF MAPS

The main objects of study are maps $f : \mathbb{X} \to \mathbb{X}$

Assume $(\mathbb{X}, \mathcal{B}, \mu)$ is a probability space.

(a) $f : \mathbb{X} \to \mathbb{X}$ is a measure preserving transformation (mpt) if it is measurable and measure preserving

(b) $f : \mathbb{X} \to \mathbb{X}$ is an invertible measure preserving transformation if it is bijective, bi-measurable and bi-measure preserving (nickname impt)

Shorthand terminology:

It is usual just to say f on $\mathbb{X}$ is a mpt with invariant measure μ and leave the σ-algebra implicit.

ORBITS

Since $f: X \to X$ we can compute $f(x), f(f(x)) = f^2(x), \ldots$

- The forward orbit of x under f is

$$O^+(x,f) = \{x, f(x), f^2(x), \ldots \}$$

If f is not invertible, we only consider forward orbits

- If f is an impt, the full orbit of x under f is

$$o(x,f) = \{\ldots, f^{-2}(x), f^{-1}(x), x, f(x), f^2(x), \ldots \}$$

where $f^{-n}(x) = (f^{-1})^n(x)$

- Question: what can we say about $o(x,f)$ for a full μ-measure set in X i.e. almost everywhere (a.e.)?

Now let's say $\mathbb{X}$ is also a compact metric space.

$x \in \mathbb{X}$ is called *recurrent* if $\exists\, n_i \to \infty$ so that $f^{n_i}(x) \to x$. The opposite of recurrent is transient.

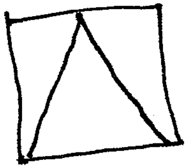
Poincaré recurrence (continuous version)

Say $f : \mathbb{X} \to \mathbb{X}$ is mpt with invariant measure μ and the σ-algebra is the Borels and f is continuous

$\Longrightarrow$ almost every point is recurrent

Proof: next lecture

Example: Tent map



is continuous, preserves Lebesgue measure and so a.e. point is recurrent