The Pointwise Ergodic Th'm, Con't

Theorem: $f : (X, \mathcal{B}, \mu) \circlearrowleft$ is a mpt and $\alpha \in L^1_{\mathbb{R}}(\mu) \Rightarrow \exists\, \alpha^* \in L^1(\mu)$ such that

$$\lim_{n \to \infty} \frac{1}{n} \cdot \sum_{i=0}^{n-1} \alpha \circ f^i(x) = \alpha^*(x) \quad \text{a.e. (i.e. pointwise)}$$

and (a) $\alpha^* \in L^1(\mu)$

(b) $\alpha^* \circ f = \alpha^*$ a.e.

(c) $\int_X \alpha^* \, d\mu = \int_X \alpha \, d\mu$

Remark: This easily implies the Theorem for $\alpha \in L^1_{\mathbb{C}}(\mu)$.

Proof:

Let $\alpha_*(x) = \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x)$

$\alpha^*(x) = \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x)$

Last time we showed $\alpha_* \circ f = \alpha_*$ and $\alpha^* \circ f = \alpha^*$

We still need to show

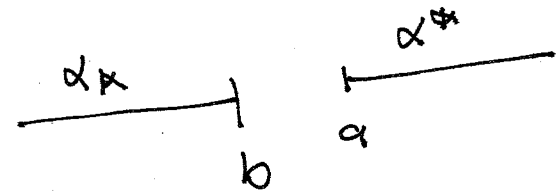
- $\alpha_* = \alpha^*$ a.e.
- $\alpha^* \in L^1(\mu)$
- $\int \alpha^* d\mu = \int \alpha \, d\mu$.

We first show $\alpha_* = \alpha^*$ a.e.

For $a, b \in \mathbb{R}$ let

$E_{a,b} = \{x : \alpha_*(x) < b \text{ and } a < \alpha^*(x)\}$

Since $\alpha_* \leq \alpha^*$



$A = \{x : \alpha_*(x) < \alpha^*(x)\}$

$= \bigcup \{E_{ab} : b < a,\ a, b \in \mathbb{Q}\}$

so this is a countable union.

we need to show $\mu(A) = 0$.

Now since we know $\alpha^* \circ f = \alpha^*$ and $\alpha_* \circ f = \alpha_*$

we have $f^{-1}(E_{a,b}) = E_{a,b}$

Let $B_a = \{ x \in X : \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) > a \}$

then $E_{a,b} \cap B_a = E_{ab}$ from the definitions

Now apply the concrete maximal ergodic theorem which yields

$$\int_{E_{a,b} \cap B_a} \alpha \, d\mu \geq a \, \mu(E_{a,b} \cap B_a)$$

but $E_{a,b} \cap B_a = E_{ab}$ so we get

$$\int_{E_{a,b}} \alpha \, d\mu \geq a \, \mu(E_{a,b})$$

Now a trick: Do the same construction with $\alpha \to -\alpha$, $a \to -a$, $b \to -b$ and since $(-\alpha)^* = -\alpha_*$ and $(-\alpha)_* = -\alpha^*$ we get

$$\int_{E_{a,b}} \alpha \, d\mu \leq b\,\mu(E_{ab})$$

Thus

$$a\,\mu(E_{a,b}) \leq \int_{E_{a,b}} \alpha \, d\mu \leq b\,\mu(E_{a,b})$$

but recall from the definition of A above only considers $b < a$. For these it must be that $\mu(E_{ab}) = 0$ and so since A is a countable union, $\mu(A) = 0$ so $\alpha_* = \alpha^*$ a.e.

Next, to prove $\alpha^* \in L^1(\mu)$ we need a lemma from Analysis. It is sometimes part of Fatou's lemma or a corollary to it

Fatou Lemma: Say $\{g_n\}, g$ are non-negative measurable functions and $g_n \to g$ a.e. pointwise and $\int g_n \, d\mu \le M$ for all $n \Rightarrow \int g \, d\mu \le M$

For the application, let $g_n = \left| \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i \right|$ and so we have just shown that $g_n \to |\alpha^*|$ pointwise, a.e.. Now note

$$\int g_n \, d\mu \le \frac{1}{n} \sum_{i=0}^{n-1} \int |\alpha \circ f^i| \, d\mu =$$

$$= \frac{1}{n} \sum_{i=0}^{n-1} \int |\alpha| \, d\mu = \int |\alpha| \, d\mu := M < \infty$$

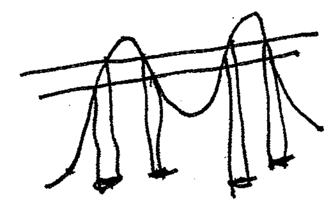
since $\alpha \in L^1(\mu)$

change of variables

Thus by Fatou

$$\int |\alpha^*| \, d\mu \leq M \quad \text{so } \alpha^* \in L^1(\mu)$$

The last thing to show is that $\int \alpha \, d\mu = \int \alpha^* \, d\mu$



For $k \in \mathbb{Z}$, $n \geq 1$ let

$$D_k^n = \left\{ x \in X : \frac{k}{n} \leq \alpha^*(x) < \frac{(k+1)}{n} \right\}$$

Define $B_{\frac{k}{n}-\varepsilon} = \left\{ x \in X : \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) > \frac{k}{n} - \varepsilon \right\}$.

By the concrete version of the maximal ergodic theorem since $f^{-1}(D_k^n) = D_k^n$

$$\int_{D_k^n \cap B_{\frac{k}{n}-\varepsilon}} \alpha \, d\mu \geq \left(\frac{k}{n} - \varepsilon\right) \mu\left(D_k^n \cap B_{\frac{k}{n}-\varepsilon}\right)$$

but when ε is small $D_k^n \wedge B_{\frac{k}{n}-\varepsilon} = D_k^n$

and so
$$\int_{D_k^n} \alpha \, d\mu \geq \left(\frac{k}{n} - \varepsilon\right) \mu(D_k^n).$$

This is true for all small epsilon and so
$$\int_{D_k^n} \alpha \, d\mu \geq \frac{k}{n} \mu(D_k^n)$$

Since $\alpha^* < \frac{k+1}{n}$ on D_k^n
$$\int_{D_k^n} \alpha^* \, d\mu \leq \frac{k+1}{n} \mu(D_k^n) = \frac{k}{n} \mu(D_k^n) + \frac{1}{n} \mu(D_k^n)$$
$$\leq \int_{D_k^n} \alpha \, d\mu + \frac{1}{n} \mu(D_k^n)$$

Now sum over k using $\bigcup_{k \in \mathbb{Z}} D_k^n = X$ by construction as a disjoint union.

yields

$$\int_X \alpha^* d\mu \leq \frac{\mu(X)}{n} + \int \alpha \, d\mu$$

$$= \frac{1}{n} + \int \alpha \, d\mu$$

Now this is true for all $n \geq 1$ and so

$$\int \alpha^* d\mu \leq \int \alpha \, d\mu$$

Apply this to $-\alpha$ yields but recall $(-\alpha)^* = -\alpha_*$

$$\int (-\alpha)^* d\mu \leq \int -\alpha \, d\mu$$

so $-\int \alpha_* d\mu \leq -\int \alpha \, d\mu$

and so $\int \alpha_* d\mu \geq \int \alpha \, d\mu$.

but $\alpha_* = \alpha^*$ ae and so $\int \alpha^* d\mu = \int \alpha \, d\mu$

since $\int \alpha^* d\mu = \int \alpha_* d\mu$ $\blacksquare$

COR $f:(X,\mathcal{B},\mu)\circlearrowleft$ is a mpt. f is ergodic $\Leftrightarrow$ $\forall \alpha \in L^1(\mu)$, $\frac{1}{n}\sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$ a.e.

Proof We already showed $\Rightarrow$. For the complement Let $A \in \mathcal{B}$ with $f^{-1}(A)=A$ and consider $\chi_A \in L^1(\mu)$. By assumption

$$\frac{1}{n}\sum_{i=0}^{n-1} \chi_A(f^i(x)) \to \int \chi_A \, d\mu = \mu(A) \text{ a.e.}$$

Now since $f^{-1}(A)=A$ if $x \in A$, $\chi_A(f^i(x)) = 1$ for all i and if $x \notin A$, $\chi_A(f^i(x)) = 0$ for all i

Thus $\mu(A) = 0$ or 1 as required for ergodicity. $\blacksquare$