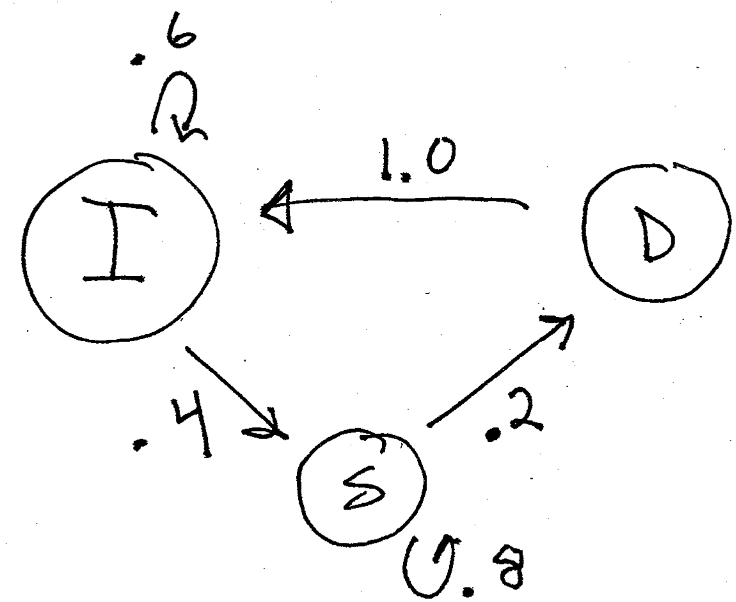
Markov chains and measures cont.

EG: 3 compartment disease

I = Immune S = suseptable D = Diseased



$$
\begin{array}{c|ccc}
 & I & S & D \\
\hline
I & .6 & .4 & 0 \\
S & 0 & .8 & .2 \\
D & 1.0 & 0 & 0
\end{array}
$$

Step size = 1 week

NOTES: . Recovery from disease yields immunity

. Immunity not permanent

(numbers are made up.)

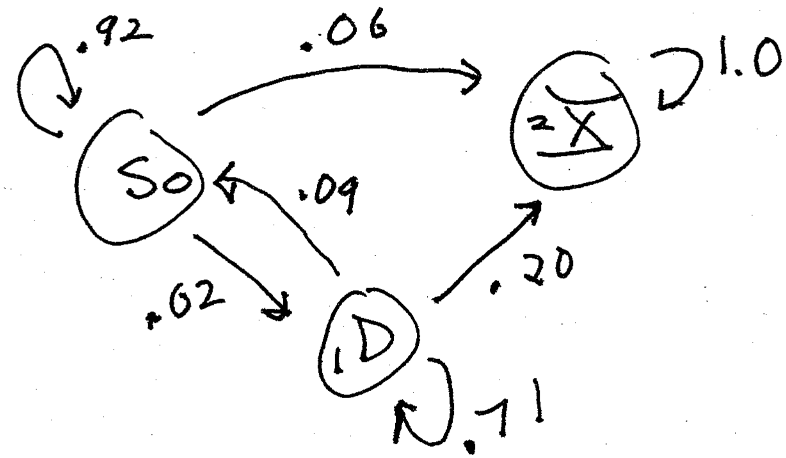
E.G.: Disease with death, no immunity

"A primer in the application of Markov Chains to wildlife disease dynamics"; Methods in Ecology and Evolution, 04 May 2010

House finch pathogen, ½-week time step

$\bar{X}$ = Death

S = Suseptable, D = Disease,



$$A = \begin{array}{c|c} & \begin{array}{ccc} S & D & \bar{X} \end{array} \\ \begin{array}{c} S \\ D \\ \bar{X} \end{array} & \begin{bmatrix} .92 & .02 & .06 \\ .09 & .71 & .20 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

$\bar{X}$ is an absorbing state.

DEF: A is a <u>stochastic matrix</u> if $A_{ij} \geq 0$ for all i,j and Row sums are all 1, or $\sum_{j=1}^{n} A_{ij} = 1$ for all i.

FACT: Given a stochastic matrix A, there always exists at least one left eigenvector with eigenvalue 1, so $\vec{p}A = \vec{p}$, and in addition $p_i \geq 0$ and $\sum p_i = 1$ (see below)

DEF: Given an $n \times n$ matrix A with $\vec{p}$ as above the corresponding $(\vec{p}, A)$ Markov measure is defined on cylinder sets ~~by~~ in Σ_n by

$$\mu\left({}_m[b_0 b_1 \dots b_k]\right) = \vec{p}_{b_0} A_{b_0 b_1} A_{b_1 b_2} \cdots A_{b_{k-1} b_k}$$

NOTE: We index $\vec{p}$ and A with $0, \dots, n-1$ since $\Sigma_n = \{0, \dots, n-1\}^{\mathbb{Z}}$

<u>FACT</u>: A $(\vec{p}, A)$ Markov measure defines a σ-invariant Borel measure on Σ_n

<u>Proof</u>: We first check basic additivity on cylinder sets

(1) $${}_m[b_0 b_1 \ldots b_k] = \coprod_{j=0}^{n-1} {}_m[b_0 b_1 \ldots b_k j]$$

disjoint union

This requres

$$\mu({}_m[b_0 b_1 \ldots b_k]) = \sum_{j=0}^{n-1} \mu({}_m[b_0 \ldots b_k j])$$

or

$$p_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k} = \sum_{j=0}^{n-1} p_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k} A_{b_k j}$$

$$= p_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k} \sum_{j=0}^{n-1} A_{b_k j}$$

$$= p_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k} \cdot 1$$

Since Row Sums are one.

eg $${}_2[01] = {}_2[010] \amalg {}_2[011] \text{ in } \Sigma_2.$$

(2) $${}_m[b_0 \dots b_k] = \coprod_{j=0}^{n-1} {}_{m-1}[j\, b_0 \dots b_k]$$

or $$P_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k} = \sum_{j=0}^{n-1} P_j A_{j b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k}$$

$$= \left(\sum_{j=0}^{n-1} P_j A_{j b_0} \right) A_{b_0 b_1} \cdots A_{b_{k-1} b_k}$$

$$= P_{b_0} A_{b_0 b_1} \cdots A_{b_{k-1} b_k}$$

since $\vec{P} A = \vec{P}$

~~eg: in Σ_2,~~ ${}_2[10] = {}_1[010] \amalg {}_1[110]$

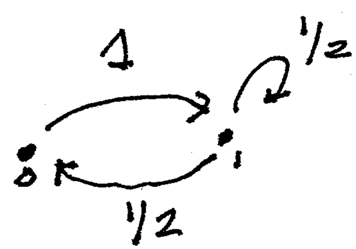
(3) μ then extends to the Borels as in earlier lectures

(4) Since the definition of $\mu({}_m[b_0 \dots b_k])$ is independent of m and $\sigma^{-1}({}_m[b_0 \dots b_k]) = {}_{m+1}[b_0 \dots b_k]$, the Markov measure is σ-invariant.

- A matrix A is non-negative, $A \geq 0$, if $A_{ij} \geq 0$ $\forall i,j$
- A non-negative matrix is <u>irreducible</u> if $\forall i,j$ $\exists n$ so that $(A^n)_{ij} > 0$

<u>Interpretation</u> - if A is the incidence matrix of a graph (so $A_{ij} > 0 \iff$ there is an edge from i to j). Then irreducibility means for every pair of vertices i and j there is some n so that there is a length n-path from i to j

$$\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$



$0 \to 0$ needs $n=2$ length path
all others length one.

- A non-negative matrix is <u>primitive</u> if $\exists N$ with $A^N > 0$ (This is also called irreducible and a-periodic)
- previous examp $\begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix} > 0$

  so $N = 2$
- Interpretation: $\exists$ a uniform N so that every pair i and j have a length N path from i to j in the graph.
- The central theorem about non-negative matrices is Perron-Fröbenius which provides the key to understanding Markov measures.

$$\begin{tikzcd} 0 \arrow[r, bend left, "1.0"] & 1 \arrow[l, bend left, "1.0"] \end{tikzcd}$$

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \quad A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \ldots$$

A is irreducible not primitive

- 3 stage disease with immunity is primitive $A^3 > 0$
- disease with death is not irreducible since no transitions out of III.

Theorem (P.F.) A is non-negative $n \times n$ matrix

(1) $\exists$ eigenvalue $\lambda \geq 0$ so that for every other eigenvalue η, $|\eta| \leq \lambda$

(2) Corresponding to λ there is a left eigenvector $\vec{u}$ with $\vec{u} \geq 0$ and a right eigenvector $\vec{v} \geq 0$

(3) If A is irreducible $\Rightarrow$ λ is a simple eigenvalue and $\vec{u} > 0$ and $\vec{v} > 0$

(4) If A is irreducible $\Rightarrow$ λ is the only eigenvalue of A with a non-negative eigenvector.

(5) If A is primitive $\Rightarrow$

$$\lim_{k \to \infty} \frac{(A^k)_{ij}}{\lambda^k} = u_j v_i$$

$\boxed{\text{Application to stochastic matrices}}$

- Assume now $A \geq 0$ is stochastic, Thus it has row sums $= 1$
- It follows that $A\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$, so $\vec{1} = \begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}$

is a right eigenvector for the eigen value 1.

- Now recall the ∞-norm of a matrix (induced by the ∞ norm on vectors) is the maximum row sum (when $A \geq 0$)

Thus $\|A\|_\infty = 1$ and so $\|A\vec{v}\| \leq \|A\|\,\|v\| \leq \|v\|$

Thus if $A\vec{v} = \eta\vec{v} \Rightarrow |\eta| \leq 1$

- Thus 1 is the P.F. eigenvalue of A
- Now assume A is irreducible, then by PF there is a unique left eigen vector $\vec{p} > 0$ for the eigenvalue 1, or $\vec{p}A = \vec{p}$

- Given an irreducible, stochastic A, this unique $\vec{p}$ is the one we choose to build the Markov measure on Σ_n.

- If A is also primitive, since $\lambda = 1$ and the left eigenvector is $[p_0\ p_1 \ldots p_{n-1}]$ and the right eigenvector is $\vec{1} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$

By part (5) of PF

$$\lim_{k \to \infty} (A^k)_{ij} = u_j v_i$$

So

$$A^k \to \begin{bmatrix} \vec{p} \\ \vec{p} \\ \vdots \\ \vec{p} \end{bmatrix}$$

__Theorem__: • Given an irreducible, stochastic matrix A and its unique left eigenvector $\vec{p}$ with $\vec{p} > 0$ and $\sum p_i = 1$ the Markov measure constructed from $(\vec{p}, A)$ is ergodic.

• If in addition, A is primitive $\Rightarrow$ the Markov measure is mixing.