A excample for last time's result

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "1"] \end{tikzcd}$$

A is irreducible $\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{odd} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{even} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

but not primitive

$\lim\limits_{n \to \infty} A^n$ doesn't exist

but $\lim\limits_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} A^i = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

The only positive measure sets under the Markov measure are $01.0101\ldots$ and 10.101010 – a period 2 orbit whose invariant measure assigns $1/2$ to each point for a total mass of one.

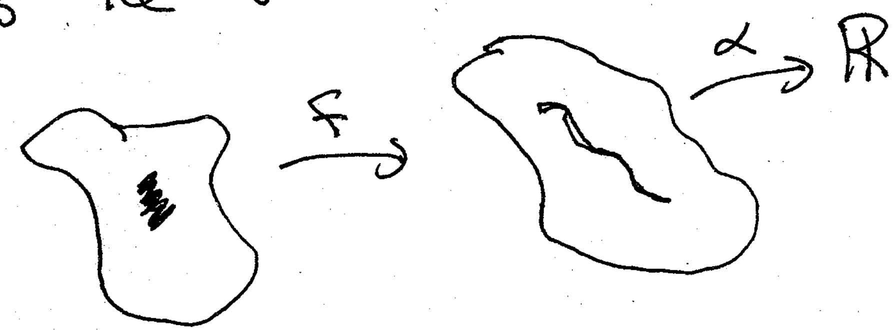
The Koopman operator and the mean (von Neumann) Ergodic Theorem

$f : (X, \mathcal{B}, \mu) \circlearrowleft$ is a mpt of a probability space. The Koopman operator (which we defined but didn't name) is

$$U : L^p(\mu) \circlearrowleft \quad \text{via} \quad U(\alpha) = \alpha \circ f$$

we showed that it's image is in $L^p(\mu)$ and $\|U\|_p = 1$ using the change of variables formula.

U tracks the evolution of scalars



- For example, if A represents a patch of dye χ_A indicates where the dye is
- Then $U(\chi_A) = \chi_A \circ f = \chi_{f^{-1}(A)}$
- so $U(\chi_A)$ indicates where the dye is after acting by f^{-1}
- [when f is bijective, and bi-measure preserving like a fluid flow, one can consider $V(\alpha) = \alpha \circ f^{-1}$ which tracks the forward evolution of scalars or observables]
- $U^n(\alpha)$ tracks evolution under $n^{\underline{th}}$ iterate
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$\boxed{\text{The } L^p\text{-mean ergodic theorem}}$ $1 \leq p < \infty$

$f : (X, \mathcal{B}, \mu) \circlearrowleft$ is mpt of a probability space.

If $\alpha \in L^p(\mu) \Rightarrow \exists\, \alpha^* \in L^p(\mu)$ with $\alpha^* \circ f = \alpha^*$ a.e.

and $\quad \dfrac{1}{n} \displaystyle\sum_{i=0}^{n-1} \alpha \circ f^i \longrightarrow \alpha^*$ in L^p

or $\quad \dfrac{1}{n} \displaystyle\sum_{i=0}^{n-1} U^i(\alpha) \to \alpha^*$ in L^p, where

U is the Koopman operator.

PROOF (sketch) If α is bounded $\Rightarrow \dfrac{1}{n} \displaystyle\sum_{i=0}^{n-1} \alpha \circ f^i =: S_n$ is bounded. By the pointwise ergodic theorem, $\exists\, \alpha^*$ with $S_n(x) \to \alpha^*$ a.e. and α^* is also bounded and so $\alpha^* \in L^p(\mu)$ and $S_n \to \alpha^*$ in L^p using the bounded convergence theorem. When α is unbounded, approximate by bounded functions (non-trivial!)

The L^2-ergodic theorem

- This was the original version of von Neumann's
- It is essentially linear algebra (infinite dimensional)
- It concerns operators with $\|U\| \leq 1$.
- The application is for $U(\alpha) = \alpha \circ f$ $U: L^2(\mu) \circlearrowleft$
- The application has $\|U\| = 1$
- $\mathcal{H}$ denotes a general Hilbert space with complex valued inner product $\langle \alpha, \beta \rangle$
- The concrete application in $L^2(\mu)$ has $\langle \alpha, \beta \rangle = \int_X \alpha \bar{\beta} \, d\mu$
- If $U: \mathcal{H} \circlearrowleft$ is a linear operator, its <u>adjoint</u> is the unique linear operator with $\langle U^* \alpha, \beta \rangle = \langle \alpha, U \beta \rangle$ $\forall \alpha, \beta \in \mathcal{H}$. (If $\mathcal{H} = \mathbb{R}^n$, $\langle \vec{\alpha}, \vec{\beta} \rangle = \vec{\alpha} \cdot \vec{\beta} \Rightarrow U^* = U^T$)

Theorem $U:\mathcal{H}\mathcal{H}$ is a Linear operator with $\|U\| \le 1$.

$\mathcal{M} = \{\alpha : U\alpha = \alpha\}$ and $P:\mathcal{H}\to\mathcal{M}$ is orthogonal projection

Then for each $\alpha \in \mathcal{H}$

$$\frac{1}{n}\sum_{i=0}^{n-1} U^i \alpha \longrightarrow P\alpha$$

Remarks • P is projection onto invariant functions ($U(\alpha) = \alpha \circ f$) The

• Letting $P\alpha = \alpha^*$ (so $\alpha^* f = \alpha^*$ since

Theorem says $\left\| \frac{1}{n}\sum_{i=0}^{n-1} \alpha \circ f^i \to \alpha^* \right\|_2$

Examples (1) $U:\mathbb{R}^2 \to \mathbb{R}^2$ via $U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ Rigid Rotation

has $\|U\| = 1$, $U\vec{x} = \vec{x} \Leftrightarrow \vec{x} = 0$ and $\forall \vec{x}$

$$\frac{1}{n}\sum_{i=0}^{n-1} U^i \vec{x} \to 0$$

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- $U: \mathbb{R}^2 \to U\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $\quad U\vec{x} = \vec{x} \Leftrightarrow$ $\vec{x} \in \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. For $\vec{y} = \begin{pmatrix} a \\ b \end{pmatrix}$,

$$\frac{1}{n} \sum_{i=0}^{n-1} U^i \begin{pmatrix} a \\ b \end{pmatrix} \to \begin{pmatrix} a/2 + b/2 \\ a/2 + b/2 \end{pmatrix} \in \operatorname{span}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

so $P\begin{pmatrix} a \\ b \end{pmatrix} = \begin{matrix} (a+b)/2 \\ (a+b)/2. \end{matrix}$

Proof of Theorem: $\mathcal{M}$ is a closed subspace and let $\mathcal{N}$ be the closure of the linear subspace $\{\alpha - U\alpha : \alpha \in \mathcal{H}\}$. We show first that

$\mathcal{M} = \mathcal{N}^{\perp} = \{\alpha \in \mathcal{H} : \langle \alpha, \beta \rangle = 0 \;\; \forall \beta \in \mathcal{N}\}$

This implies that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ as orthogonal decomposition by direct sum.

Say $\alpha \in \mathcal{M}^\perp$ which implies $\langle \alpha, \beta - U\beta \rangle = 0$

for all $\beta \in \mathcal{H}$. Thus

$$0 = \langle \alpha, \beta \rangle - \langle \alpha, U\beta \rangle = \langle \alpha, \beta \rangle - \langle U^*\alpha, \beta \rangle$$

$$= \langle \alpha - U^*\alpha, \beta \rangle$$

This holds for all $\beta \in \mathcal{H}$ and so $\alpha - U^*\alpha = 0$ or $U^*\alpha = \alpha$, but what we want is $U\alpha = \alpha$ note that so that $\mathcal{M}^T \subseteq \mathcal{M}$. To prove that

$$\|U\alpha - \alpha\|^2 = \langle U\alpha - \alpha, U\alpha - \alpha \rangle = \langle U\alpha, U\alpha \rangle - \langle U\alpha, \alpha \rangle - \langle \alpha, U\alpha \rangle + \langle \alpha, \alpha \rangle$$

$$= \|U\alpha\|^2 - \langle \alpha, U^*\alpha \rangle - \langle U^*\alpha, \alpha \rangle + \|\alpha\|^2$$

$$= \|U\alpha\|^2 - \langle \alpha, \alpha \rangle - \langle \alpha, \alpha \rangle + \|\alpha\|^2$$

$$\leq \|\alpha\|^2 - 2\|\alpha\|^2 + \|\alpha\|^2 = 0$$

$\|U\alpha - \alpha\| = 0$ or $U\alpha = \alpha$ so $\mathcal{M}^T \subseteq \mathcal{M}$

So

Conversely, if $\alpha \in \mathfrak{M} \Rightarrow U\alpha = \alpha$ and applying the argument we just gave $(U^{**} = U)$ we have

$U^*\alpha = \alpha$ after we show that $\|U^*\| \leq 1$. To

prove that $\|U^*\alpha\|^2 = |\langle U^*\alpha, U^*\alpha\rangle| = |\langle UU^*\alpha, \alpha\rangle|$

$\leq \|UU^*\alpha\| \, \|\alpha\| \leq \|U^*\alpha\| \, \|\alpha\|$

and so $\|U^*\alpha\| \leq \|\alpha\|$ which means $\|U^*\| \leq 1$

[Along the way we used the Cauchy-Swartz inequality which holds in any Hilbert space $|\langle\alpha, \beta\rangle| \leq \|\alpha\| \, \|\beta\|$]

Now using $U\alpha = \alpha$ implies $U^*\alpha = \alpha$ we get for all $\beta \in \mathcal{H}$

$$\langle\alpha, \beta - U\beta\rangle = \langle\alpha, \beta\rangle - \langle\alpha, U\beta\rangle$$
$$= \langle\alpha, \beta\rangle - \langle U^*\alpha, \beta\rangle$$
$$= \langle\alpha - U^*\alpha, \beta\rangle = \langle 0, \beta\rangle = 0$$

and so $\alpha \in \mathfrak{N}^\perp$ and so $\mathfrak{M} \subseteq \mathfrak{N}^\perp$ and thus

$$\mathfrak{M} = \mathfrak{N}^\perp$$

This means that $\mathcal{H} = \mathcal{M} \oplus \mathcal{N}$ an orthogonal decomposition.

Thus if $\alpha \in \mathcal{H}$, there is a unique $\alpha_0 \in \mathcal{N}$ with

$\alpha = \alpha_0 + P\alpha$ where recall $P : \mathcal{H} \to \mathcal{M}$ is orthogonal projection

We will show (a) $\alpha_0 \in \mathcal{N} \Rightarrow \frac{1}{n}\sum_{i=0}^{n-1} U^i \alpha_0 \to 0$ in L^2

(b) $\beta \in \mathcal{M} \Rightarrow \frac{1}{n}\sum_{i=0}^{n-1} U^i \beta \to \beta$ in L^2

Putting these together $\frac{1}{n}\sum_{i=0}^{n-1} U^i \alpha = \frac{1}{n}\sum_{i=0}^{n-1} U^i(\alpha_0 + P\alpha) \to P\alpha$, as required.

(b) is easy, since $\beta \in \mathcal{M}$, $U\beta = \beta$ so $U^i\beta = \beta$

and so $\frac{1}{n}\sum_{i=0}^{n-1} U^i \beta = \frac{1}{n} \cdot n\beta = \beta$ for all n.

Now for (a). Say $\alpha_0 = \gamma - U\gamma$ i.e. a generator of M

then $\frac{1}{n}\sum_{i=0}^{n-1} U^i(\alpha_0) = \frac{1}{n}\left[(\gamma - U\gamma) + (U\gamma - U^2\gamma) + \dots + U^{n-1}\gamma - U^n\gamma\right]$

$$= \frac{1}{n}\left[\gamma - U^n\gamma\right]$$

Thus $\left\| \frac{1}{n}\sum_{i=0}^{n-1} U^i(\alpha_0) \right\| \le \frac{1}{n}\left(\|\gamma\| + \|U^n\gamma\|\right)$

$\le \frac{1}{n}\left(\|\gamma\| + \|\gamma\|\right) \to 0$

$\|U\| \le 1 \Rightarrow \|U^n\| \le 1 \Rightarrow \|U^n\gamma\| \le \|U^n\|\,\|\gamma\| \le \|\gamma\|$.

We now deal with closure:

Say $\exists\ \gamma_j$ so that $\alpha_j := \gamma_j - U\gamma_j \to \alpha_0$

By the triangle inequality $\left\|\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_0\right\| \le \left\|\frac{1}{n}\sum_{i=0}^{n-1} U^i(\alpha_0 - \alpha_j)\right\| + \left\|\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_j\right\|$ (*)

for each j.

We estimate each term in $(*)$. Given $\varepsilon > 0$

First pick j so that $\|\alpha_0 - \alpha_j\| < \varepsilon/2$

Then
$$\Big\|\frac{1}{n}\sum_{i=0}^{n-1} U^i(\alpha_0-\alpha_j)\Big\| \le \frac{1}{n}\sum_{i=0}^{n-1}\|U^i(\alpha_0-\alpha_j)\|$$
$$\le \frac{1}{n}\sum_{i=0}^{n-1}\|U^i\|\,\|\alpha_0-\alpha_j\|$$
$$< \frac{1}{n}\sum_{i=0}^{n-1}\varepsilon/2 = \varepsilon/2$$

Now for this j, pick n so that
$$\Big\|\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_j\Big\| < \varepsilon/2$$
this is possible since $\alpha_j = \gamma_j - U\gamma_j$ and so by the above $\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_j \to 0$.

Thus using $(*)$ $\Big\|\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_0\Big\| < \varepsilon$. Since ε is arbitrary, $\frac{1}{n}\sum_{i=0}^{n-1} U^i\alpha_0 \to 0$, finishing the proof. ∎