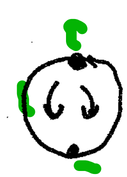
ET15

$\boxed{\text{Continuous measure preserving transformations}}$

- We now examine the interaction of dynamics, measure theory and topology.
- So X is now a separable metric space, often compact. $\mathcal{B}$ = Borel sigma algebra.
- $f : X \circlearrowleft$ preserves a Borel probability measure $(X, \mathcal{B}, \mu)$ <u>and</u> is continuous and onto.

- Important note: A given continuous $f : X \circlearrowleft$ can preserve many (even uncountably many) different Borel measures

- Example: Bernoulli measures on (Σ_n, σ)
- Example:  exactly 2 invariant measures.

- Recall that a separable metric space has a countable base for its topology $\{U_j\}$

- DEF: $Z \subseteq X$ is dense $\Leftrightarrow$ $\overline{Z} = X$ (overbar = closure)
- FACT: Z is dense $\Leftrightarrow$ $Z \cap U_j \neq \emptyset$ $\forall j$

- <u>Theorem</u>: Assume $f : (X, \mathcal{B}, \mu) \circlearrowleft$ is a continuous, onto map of a compact metric space and μ is an invariant Borel probability measure

  (a) a.e. point $x \in X$ is positively recurrent ($\exists\, n_i \to \infty$ with $f^{n_i}(x) \to x$)

  (b) If f is ergodic and μ is positive on open sets $\Rightarrow$ a.e. point has a dense forward orbit ($\overline{O^+(x, f)} = X$)

<u>Terminology</u>: If f has a dense orbit, it is called <u>transitive</u>.

<u>Proof</u> We proved (a) earlier in the semester.

For (b), fix j and let Z_j be the set of all $x \in X$ with $\lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \chi_{U_j}(f^i(x)) \to \int \chi_{U_j}\, d\mu = \mu(U_j) > 0$.

Since f is ergodic, $\mu(Z_j) = 1$. Now since $\mu(U_j) > 0$, for some i (in fact infinitely many i), $\chi_{U_j}(f^i(x)) = 1$ or $f^i(x) \in U_j$. Finally, let $Z = \bigcap Z_j$. Then $\mu(Z) = 1$ and $x \in Z \Rightarrow o^+(x,f) \cap U_j \neq \emptyset$ for all j so $\overline{o^+(x,f)} = X$ ∎

<u>Example</u>: We will see below that a Bernoulli or Markov measure with all $p_i > 0$ is positive on open sets.

- Many measures are not positive on open sets so we restrict attention to where they are

- Definition: The support of a measure μ on X is

$$\operatorname{spt}(\mu) = \bigcap \{ C_\alpha : \mu(C_\alpha) = 1 \text{ and } C_\alpha \text{ is closed} \}$$

$$= \left(\bigcup \{ U_\alpha : \mu(U_\alpha) = 0 \text{ and } U_\alpha \text{ is open} \} \right)^c$$

- equality is an easy argument
- Since X is cpt, C is closed $\Leftrightarrow$ C is cpt.
- A def. from Topology: If $Z \subseteq X$ then the induced topology on subspace topolog is defined to have V open $\Leftrightarrow$ $\exists$ open U in X such that $V = U \cap Z$
- $\therefore$ When there is a metric, the restriction of the metric to Z yields the subspace topology

FACTS:

(1) $\mu(\mathrm{spt}(\mu))=1$ and $\mathrm{spt}(\mu)$ is compact

(2) If $V \subseteq \mathrm{spt}(\mu)$ is open in the subspace topology on $\mathrm{spt}(\mu) \Rightarrow \mu(V)>0$. Proof: V is open in the subspace topology so $\exists$ open $U \subseteq X$ with $V = U \cap \mathrm{spt}(\mu)$. For the sake of contradiction say $\mu(V)=0$. Now $U = V \amalg U-\mathrm{spt}(\mu)$ implies $\mu(U)=\mu(V)+\mu(U-\mathrm{spt}(\mu))$. Now since $\mathrm{spt}(\mu)$ is closed, $U-\mathrm{spt}(\mu)$ is open and in $(\mathrm{spt}(\mu))^c$ and so $\mu(U-\mathrm{spt}(\mu))=0$ and so $\mu(U)=0$ so $U \cap \mathrm{spt}(\mu)=\emptyset$, a contradiction.

(3) If f is mpt, $B \in \mathcal{B}$ since $B \subseteq f^{-1}(f(B))$ we have $\mu(f(B)) = \mu(f^{-1}(f(B))) \geq \mu(B)$ and so $\mu(f(\mathrm{spt}(\mu)))=1$.

<u>Lemma</u>: X cpt, metric, $f:(X,\mathcal{B},\mu)\circlearrowleft$ is continuous and measure preserving, $\mathcal{B}=$ Borels

(a) $spt(\mu)$ is f-invariant $f(spt(\mu))=spt(\mu)$

(b) In the subspace topology on $spt(\mu)$ (i.e. in the restriction of the metric to $spt(\mu)$) μ restricted to $spt(\mu)$ is positive on open sets.

<u>Proof</u> (b) follows directly from FACT (2)

For (a), Assume C is cpt with $\mu(C)=1$ and

Thus $f^{-1}(C)$ is closed and $\mu(f^{-1}(C))=\mu(C)=1$.

Thus from the definition of $spt(\mu)$, $spt(\mu)\subseteq f^{-1}(C)$

and so $f(spt(\mu))\subseteq f(f^{-1}(C))=C$ for every closed C with $\mu(C)=1$ and so $f(spt(\mu))\subseteq$

$\bigcap\{C : C \text{ is closed}, \mu(C)=1\}=spt(\mu)$

On the other hand, $spt(\mu)$ is compact and thus $f(spt(\mu))$ is cpt and thus closed and by FACT(3), $\mu(f(spt(\mu))) = 1$. Thus by the def of $spt(\mu)$, $spt(\mu) \subseteq f(spt(\mu))$ and $spt(\mu) = f(spt(\mu))$ $\blacksquare$

Remark: If f is not injective, then usually $f^{-1}(spt(\mu))$ is larger than $spt(\mu)$.

Example is a pre periodic point or even

$f(b)=b$ $f(a)=b$ The only invariant measure is $\mu(b)=1$ and $spt(\mu)=\{b\}$ and $f^{-1}(spt(\mu))=\{a,b\}$

COR: With f, $(X, \mathcal{B}, \mu)$ as above

(a) The support of every Borel, invariant probability measure is contained in the closure of the set of recurrent points of

(b) If f is ergodic, then for a.e. x,

$$\overline{O^+(x,f)} = \operatorname{spt}(\mu).$$

Proof (a) We know that a.e. point is recurrent, so let Z be a full measure set with respect to μ with $\mu(Z)=1$. Since $\mu(\operatorname{spt}(\mu))=1$, $Z' = Z \cap \operatorname{spt}(\mu)$ has $\mu(Z')=1$. Now we claim $\overline{Z'} = \operatorname{spt}(\mu)$ ($\overline{Z'}$ = closured Z') if not, $U = \operatorname{spt}(\mu) - \overline{Z'}$ is open, has zero measure and is in $\operatorname{spt}(\mu)$, a contradiction. Since $Z' \subseteq$ set of recurrent points

$$\operatorname{spt}(\mu) = \overline{Z'} \subseteq \overline{\{\text{set of recurrent points}\}},$$ finishing (a)

For (b) Since $\operatorname{spt}(\mu)$ is compact and f invariant we can apply the Theorem part (b) to $f|_{\operatorname{spt}(\mu)}$. ∎

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Markov shifts and measures

- We need a topology on $\Sigma_n = \{0, \ldots, n-1\}^{\mathbb{Z}}$
- The simplest topology is the product
- It is generated by many metrics, a simple one (different than we used last semester) is
  $d(\underline{s}, \underline{s}') = \frac{1}{2^\ell}$ where $\ell = \min\{|i| : x_i \neq x_i'\}$
- Thus if $d(\underline{s}, \underline{s}') < \frac{1}{2^\ell}$, $s_i = s_i'$ for all $|i| \leq \ell$
- So $B_{1/2^\ell}(\underline{s}) = [s_{-\ell} \ldots s_{-1} s_0 s_1 \ldots s_\ell]_{-\ell}$
- Informally, $\underline{s}$ and $\underline{s}'$ are close $\iff$ the agree in a big middle chunk.

<u>FACTS</u> (1) The collection of all cylinder sets form a base for the topology

(2) The left shift $\sigma : \Sigma_n \circlearrowleft$ is a homeomorphism whose inverse is the right shift.

<u>Lemma</u>: Markov and Bernoulli measures with $\vec{p} > 0$ A > 0 are positive on open sets and ergodic. Thus a.e. point has dense orbit

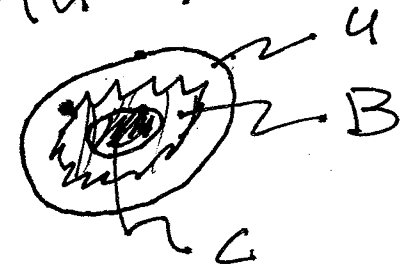
<u>Proof</u> We showed they are ergodic when $\vec{p} > 0$, for any cylinder set C, $\mu(C) > 0$, so by the above.

<u>NOTICE</u>: We have uncountably many Markov and Bernoulli measures. Since they are positive on all open sets, the support of every one of these measures is all of Σ_n. How do they interact?

Measures on metric spaces

- We need more theory first about the ~~intersect~~ interaction of topology and measure

- A *regular* measure on a metric space X is one for which $\forall B \in \mathcal{B}$ and $\varepsilon > 0$, there exists an open set U and a closed set C so that $C \subseteq B \subseteq U$ and $\mu(U - C) < \varepsilon$



- **Theorem**: Borel probability measures on metric spaces are regular.

<u>COR</u>: For a Borel probability measure on the metric space $\underline{X}$ and $B \in \mathcal{B}$

$$\mu(B) = \sup\{\mu(C) : C \text{ closed}, C \subseteq B\}$$

$$\mu(B) = \inf\{\mu(U) : U \text{ open}, U \supseteq B\}$$

A little functional analysis is also useful! which we introduce next time.