Ergodic Theory: continuous on compact spaces

- $C(X, \mathbb{R}) = \{ f : X \to \mathbb{R} \mid f \text{ is continuous} \}$
- Assume X is compact, metric
- $\|\alpha\|_0 = \max\{|\alpha(x)| : x \in X\}$ notice that the max is attained and is finite since α is continuous and X is compact.

  $d(\alpha, \beta) = \|\alpha - \beta\|_0$. This makes

- $C(X, \mathbb{R})$ into a complete, separable metric space
- $C(X, \mathbb{R})$ is also a vector space over $\mathbb{R}$
- Put together, $C(X, \mathbb{R})$ is a Banach space.
- $C(X, \mathbb{R})$ is sometimes just written $C(X)$

- A Borel probability measure μ defines a Linear function $L: C(X) \to \mathbb{R}$ via.

$$L(\alpha) = \int_X \alpha \, d\mu$$

- Notice $|L(\alpha)| \leq \int_X |\alpha| \, d\mu \leq (\max|\alpha|)\, \mu(X) = \|\alpha\|_0$

- A bounded Linear functional is one for which $\exists M$ with $|L(\alpha)| \leq M \|\alpha\|_0$, so μ defines a bounded linear functional.

- The next Lemma says that the functionals associated with the measure characterize the measure.

- DEF: Let $\underline{\underline{M(X)}}$ be all Borel, probability measures on X.

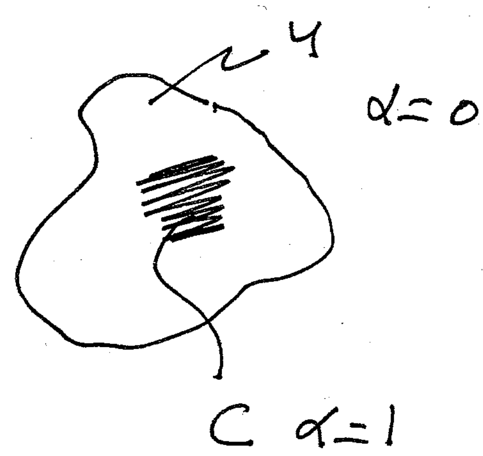
Lemma $\mu, \nu \in M(X)$. $\int \alpha \, d\mu = \int \alpha \, d\nu \ \forall \alpha \in C(X) \Leftrightarrow \mu = \nu$

Proof ($\Leftarrow$) is trivial. For ($\Rightarrow$) Recall from last time that $\mu(B) = \sup\{\mu(C) : C \text{ closed}, C \subseteq B\}$ when $B \in \mathcal{B}$. Thus it suffices to show that $\mu(C) = \nu(C)$ for all closed $C \subseteq X$.

Given $\varepsilon > 0$, by regularity of the measure μ, $\exists$ open U with $C \subseteq U$ and $\mu(U - C) < \varepsilon$

Now define $\alpha : X \to \mathbb{R}$ via.

$$\alpha(x) = \begin{cases} 0 & x \notin U \\ \dfrac{d(x, U^c)}{d(x, U^c) + d(x, C)} & x \in U \end{cases}$$



Now the denomenator is never zereo when $x \in U$. Since the distance function to a closed set is continuous, α is continuous. Finally $0 \leq \alpha \leq 1$ and $\alpha = 0$ on U^c $\alpha = 1$ on C

Thus $\mu(C) \underset{\substack{\alpha = 1 \text{ on } C \\ \alpha \geq 0}}{\leq} \int_X \alpha \, d\mu \underset{\text{by hypothes}}{=} \int_X \alpha \, d\nu \underset{\substack{0 \leq \alpha \leq 1 \\ \text{on } U}}{\leq} \nu(U) < \nu(C) + \varepsilon$

True for all ε, and so $\mu(C) \leq \nu(C)$. Reversing the roles, $\nu(C) \leq \mu(C) \Rightarrow \mu(C) = \nu(C)$ $\blacksquare$

Remark:

- So writing L_μ for $L_\mu(\alpha) = \int_X \alpha \, d\mu$ we have

$$L_\mu(\alpha) = L_\nu(\alpha) \quad \forall \alpha \in C(X) \Rightarrow \mu = \nu.$$

- The Riesz Representation Theorem says that all normalized, positive linear functions can be constructed this way

- **Theorem**: X is compact, metric, the linear functional $L : C(X) \to \mathbb{R}$ is positive ($\alpha \geq 0 \Rightarrow L(\alpha) \geq 0$), normalized ($L(1) = 1$) and bounded $\Rightarrow \exists \mu \in M(X)$ with

$$L(\alpha) = \int_X \alpha \, d\mu \quad \forall \alpha \in C(X, \mathbb{R})$$

- The weak* topology on $M(X)$ is the smallest topology making each of the maps $\Phi_\alpha : \mu \mapsto \int \alpha \, d\mu$ continuous.

  (so $\Phi_\alpha : M(X) \to \mathbb{R}$)

- Alternative, more concrete characterizations

  $\boxed{X \text{ is cpt, metric}}$

  (1) $\mu_n \to \mu$ in $M(X) \iff \forall \alpha \in C(X)$, $\int \alpha \, d\mu_n \to \int \alpha \, d\mu$

  (2) $\forall A \in \mathcal{B}$ with $\mu(Fr(A)) = 0$, $\mu_n(A) \to \mu(A)$

Remark on (2): $Fr(A)$ is topological frontier. $A \in \mathcal{B}$ with $\mu(Fr(A)) = 0$ is called a continuity set

(3) Let $\{\alpha_n\}$ be a dense subset of $C(X)$ [Recall $C(X)$ is separable)

then

$$D(\mu,\nu) = \sum_{n=1}^{\infty} \frac{|\int \alpha_n d\mu - \int \alpha_n d\nu|}{2^n \, \|\alpha_n\|_0}$$

is a metric on $M(X)$)

<u>example</u>: $X = \Sigma_n$, $\{\chi_C : C \text{ is a cylinder set}\}$ is dense in $C(\Sigma_n, \mathbb{R})$. Since $\int \chi_C \, d\mu = \mu(C)$

we have $D(\mu,\nu) = \sum_{n=1}^{\infty} \frac{|\mu(C_n) - \nu(C_n)|}{2^n}$

where $\{C_n\}$ is an enumeration of the cylinder sets of Σ_n

If μ and ν are invariant measures under σ,

Since $\sigma^{-n}({}_k[B]) = {}_{k+n}[B]$ and so

$\mu(\sigma^{-n}({}_k[B])) = \mu({}_{k+n}[B]) = \mu({}_k[B])$ it suffices

when restricted to invariant measures to

consider

$$D(\mu,\nu) = \sum_{n=1}^{\infty} \frac{|\mu({}_0[B_n]) - \nu({}_0[B_n])|}{2^n}$$

with B_n an enumeration of the blocks of Σ_n

For example, in Σ_2, $0, 1, 00, 01, 10, 11, 000, \ldots$

Theorem When X is compact, metric,

$M(X)$ with the weak* topology is compact.

- DEF of the push forward of a measure
  - $f : X \to X$ is a continuous function of a compact metric space
  - Now when U is open in X, $f^{-1}(U)$ is open in X. Thus $f^{-1}(\mathcal{B}) \subseteq \mathcal{B}$
  - Define $f_* : M(X) \to M(X)$ as

$f^{-1}(B)$ B

$$f_*\mu(B) = \mu\left(f^{-1}(B)\right)$$

- By change of variables

$$\int \alpha \, d(f^*\mu) = \int \alpha \circ f \, d\mu$$
$$\forall \alpha \in C(X)$$

Theorem: $f_*: M(X) \to M(X)$ is continuous and affine.

Proof: Given $\alpha \in C(X)$ and assuming $\mu_n \to \mu$ in $M(X)$, then using change of variables and characterization (1) above

$$\int \alpha \, d f_* \mu_n = \int \alpha \circ f \, d\mu_n \to \int \alpha \circ f \, d\mu = \int \alpha \, d f_* \mu$$

since this holds for all α and so

so $f_* \mu_n \to f_* \mu$

f_* is continuous.

For affine, if $p \in [0,1]$, $f_*(p\mu + (1-p)\mu)(B)$

$= p\mu(f^{-1}B) + (1-p)\mu(f^{-1}B) = p f_*\mu(B) + (1-p) f_*\mu(B)$

holds for all $B \in \mathcal{B}$. ∎

DEF $M(X, f) = \{\mu \in M(X) : f_* \mu = \mu\}$

So $M(X, f)$ is the fixed points of f_* acting on $M(X)$.

If $\mu \in M(X, f) \Rightarrow \mu(f^{-1}(B)) = \mu(B)$ $\forall B \in \mathcal{B}$,

So $M(X, f)$ is exactly the set of invariant Borel probability measures for f.

Lemma Assume $\mu \in M(X)$. We have $\mu \in M(X, f)$

$\Longleftrightarrow \int \alpha \circ f \, d\mu = \int \alpha \, d\mu$, $\forall \alpha \in C(X)$.

Proof ($\Rightarrow$) Since $f_* \mu = \mu$, $\int \alpha \circ f \, d\mu = \int \alpha \, d f_* \mu = \int \alpha \, d\mu$

($\Leftarrow$) $\int \alpha \circ f \, d\mu = \int \alpha \, d\mu$ says that $\int \alpha \, d f_* \mu = \int \alpha \, \mu$ for all $\alpha \in C(X, \mathbb{R})$ and so by the Lemma above, $f_* \mu = \mu$ ∎