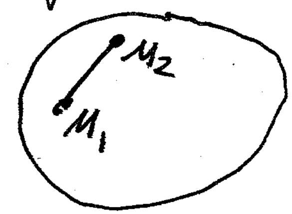
Continuous Ergodic Theory Cont.

- X is compact, metric
- $M(X)$ = all Borel, probability measures on X
- weak* Topology on $M(X)$:

  $\mu_n \to \mu_0 \iff \forall \alpha \in C(X, \mathbb{R})\ \int \alpha \, d\mu_n \to \int \alpha \, d\mu$

- There is a metric for the weak* topology.
- $M(X)$ with this topology is compact.
- $M(X)$ is also convex



Given $p \in [0,1]$, $\mu_1, \mu_2 \in M(X)$

$\Rightarrow p\mu_1 + (1-p)\mu_2 \in M(X)$.

- $f: X \to X$ onto and continuous
- Induces $f_*: M(X) \to M(X)$ via $f_*\mu(B) = \mu(f^{-1}(B))$
- f_* is continuous in weak*- Topology and affine
- $M(X,f) = \{\mu \in M(X) : f_*\mu = \mu\}$, the space of f-invariant Borel probability measures.
- $M(X,f)$ is compact and convex

Proof: $\mu_n \in M(X,f)$ with $\mu_n \to \mu$. Since f_* is continuous, $f_*\mu_n \to f_*(\mu)$ but $f_*\mu_n = \mu_n$ so $f_*(\mu) = \mu$ and so $\mu \in M(X,f)$. Since f_* is affine, $\mu_1, \mu_2 \in M(X,f) \Rightarrow f_*(p\mu_1 + (1-p)\mu_2)$ $= p f_*\mu_1 + (1-p) f_*\mu_2 = p\mu_1 + (1-p)\mu_2$ so

$$p\mu_1 + (1-p)\mu_2 \in M(X,f).$$

- We need to know that $M(X, f)$ is non-empty, i.e. There exists at least one invariant, Borel probability measure.
- This follows from applying a powerful fixed point theorem to $f_*: M(X) \to M(X)$, but a direct proof isn't hard and shows how to create invariant measures.

<u>Theorem</u> (Krylov and Bogolioubov): $f: X \circlearrowleft$ is continuous function on a compact metric space. If $\sigma \in M(X)$ define $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \sigma$ and then any limit point of $\{\mu_n\}$ is in $M(X, f)$. Further, such a limit point always exists and so $M(X, f) \neq \emptyset$.

<u>Proof</u> Note that $\mu_n \in M(X)$ since $M(X)$ is convex and since $M(X)$ is compact, $\exists$ a limit point μ, so assume $\mu_{n_j} \to \mu$.

A couple preliminaries

(1)
$$\left|\int \alpha \circ f^{n_j}\, d\sigma_n\right| \le \int |\alpha \circ f^{n_i}|\, d\sigma_n$$
$$\le \sup |\alpha \circ f^{n_i}| \int d\sigma_n$$
$$= \|\alpha\|_0$$

(2) $\int \alpha \circ f \, d f_*^i \sigma = \int \alpha \circ f^{i+1} d\sigma$ by change of variables

(3) Recall we showed above $\mu \in M(X, f)$
$\iff \int \alpha \circ f \, d\mu = \int \alpha \, d\mu \quad \forall \alpha \in C(X, \mathbb{R})$

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Pick $\alpha \in C(X, \mathbb{R})$ then

$$\left| \int \alpha \circ f \, d\mu - \int \alpha \, d\mu \right| = \lim_{j \to \infty} \left| \int \alpha \circ f \, d\mu_{n_j} - \int \alpha \, d\mu_{n_j} \right| \quad [\mu_{n_j} \to \mu]$$

$$= \lim_{j \to \infty} \left| \frac{1}{n_j} \int \alpha \circ f \, d\left[\sum_{i=0}^{n_j - 1} f_*^i \sigma \right] - \int \alpha \, d\left[\sum_{i=0}^{n_j - 1} f_*^i \sigma \right] \right| \quad [\text{using } (2)]$$

$$= \lim_{j \to \infty} \left| \frac{1}{n_j} \int \left(\sum_{i=0}^{n_j - 1} \alpha \circ f^{i+1} - \sum_{i=0}^{n_j - 1} \alpha \circ f^i \right) d\sigma \right| \quad [\text{telescoping sum}]$$

$$= \lim_{j \to \infty} \left| \frac{1}{n_j} \int (\alpha \circ f^{n_j} - \alpha) \, d\sigma \right| \quad [\Delta \neq]$$

$$\leq \lim_{j \to \infty} \frac{1}{n_j} \left(\left| \int \alpha \circ f^{n_j} \, d\sigma \right| + \left| \int \alpha \, d\sigma \right| \right) \quad [(1)]$$

$$\leq \lim_{j \to \infty} \frac{1}{n_j} 2 \|\alpha\|_0 = 0$$

<u>Remark</u> The simplest measures to work with are point masses or δ-functions. For $x \in X$ define δ_x as

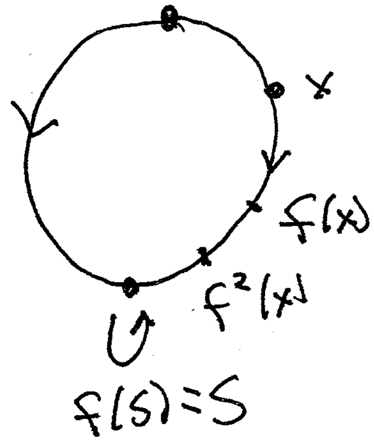
$$\delta_x(B) = 1 \text{ if } x \in B$$
$$= 0 \text{ if } x \notin B$$

Then $f_* \delta_x = \delta_{f(x)}$ and

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i \delta_x = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x)}$$

$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$

So μ_n averages point masses along an orbit, and $\mu_{n_j} \to \mu$ is a "weak* asymptotic average"



$$\frac{1}{n}\sum_{i=0}^{n-1} \delta_{f^i(x)} \to \delta_S$$

The invariant measure of the fixed point at the South pole S.

We need a bit more measure theory. This holds in general not just on the Borels on a compact metric space.

DEF: μ is absolutely continuous w.r.t. ν, if $\nu(B)=0 \Rightarrow \mu(B)=0$. This is written $\mu \ll \nu$

Example Let $\varphi \in L^1(\nu)$, $\int_X \varphi \, d\nu = 1$, $\varphi \geq 0$ define $\mu(B) = \int_B \varphi \, d\nu$. Then $\nu(B)=0 \Rightarrow \mu(B)=0$

Radon-Nikodym Theorem: Let μ, ν be two probability measures on $(X, \mathcal{B})$. $\mu \ll \nu \iff \exists\, \varphi \in L^1(\nu)$, $\varphi \geq 0$, $\int \varphi \, d\nu = 1$ and $\mu(B) = \int_B \varphi \, d\nu$, $\forall B \in \mathcal{B}$.

Notes • φ is often written $\varphi = \frac{d\mu}{d\nu}$ and one has the heuristic $d\mu = \frac{d\mu}{d\nu} d\nu$ and so

$$\mu(B) = \int_B d\mu = \int_B \frac{d\mu}{d\nu} d\nu = \int_B \varphi \, d\nu$$

• φ is called the density. It is unique up to its L^1-class, i.e. a.e.

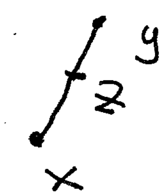
• $dm = \rho \, ds$ $\quad$ m = mass $\quad \rho$ = density = gm/cm

ds = arc length

$$\text{MASS} = \int_0^x dm = \int_0^x \rho \, ds$$

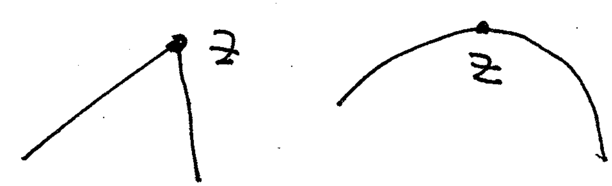
We also need a bit on convexity

Let $x, y \in$ a vector space. A <u>convex combination</u> is $px + (1-p)y$ for $p \in [0,1]$. It is called nontrivial if $p \in (0,1)$. z is said to be <u>between</u> $x \neq y$ if z is a nontrivial convex combination of x and y



- A set K <u>is convex</u> if $x, y \in K$ $\Rightarrow$ $px + (1-p)y \in K$ $\forall p \in [0,1]$ or every point between x and y is in K

- An <u>extreme point</u> of a convex set is a point z that is not between any $x, y \in K$

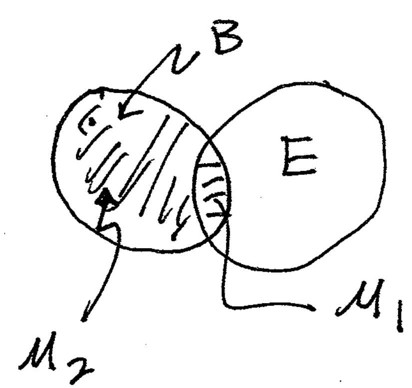


Recall from above that $M(X,f)$ is compact and convex

Theorem Assume $f: X \circlearrowleft$ is continuous map of compact metric space. $\mu \in M(X,f)$ is ergodic under $f \Leftrightarrow \mu$ is an extreme point of $M(X,f)$.

Proof ($\Leftarrow$) We prove the contrapositive so assume μ is not ergodic. Thus there exists $E \in B$ with $0 < \mu(E) < 1$ and $f^{-1}(E) = E$. Now define

$$\mu_1(B) = \frac{\mu(B \cap E)}{\mu(E)} \qquad \mu_2(B) = \frac{\mu(B \cap E^c)}{\mu(E^c)}$$



It follows that $\mu_1, \mu_2 \in M(X)$ and since $f^{-1}(E) = E$, $f_* \mu_1 = \mu_1$ and $f_* \mu_2 = \mu_2$ and $\mu_1 \neq \mu_2$. Thus $\mu_1, \mu_2 \in M(X, f)$ and certainly $\mu_1 \neq \mu_2$

$$\text{and } \mu = \mu(E)\mu_1 + \mu(E^c)\mu_2$$
$$= \mu(E)\mu_1 + (1-\mu(E))\mu_2$$

with $0 < \mu(E) < 1$ using $B = (B \wedge E) \amalg (B \wedge E^c)$.

Thus μ is not an extreme point.

$(\Rightarrow)$ next time.