Continuous Ergodic Theory, cont.

- X cpt, metric, $f: X \to$ onto, cont.
- $M(X, f)$ = all invariant, Borel probability measures.
- $M(X, f)$ in weak*-topology is compact.
- $M(X, f)$ is convex and non-empty
- $\mu \in M(X, f)$ is an extreme point if it is not on the line segment between any two $\mu_1 \neq \mu_2 \in M(X, f)$

<u>Theorem</u> μ is an extreme point $\Leftrightarrow$ it is ergodic under f

<u>Proof</u>: Last time ($\Rightarrow$)

($\Rightarrow$) Assume μ is ergodic and

$\mu = p\mu_1 + (1-p)\mu_2 \quad p \in (0,1),\ \mu_1, \mu_2 \in M(X, \mathcal{F})$.

We show that $\mu = \mu_1$ and so μ is not between any $\mu_1 \neq \mu_2$

Assuming $\mu = p\mu_1 + (1-p)\mu_2$ if $\mu(B) = 0$

Certainly $\mu_1(B) = 0$ and so $\mu_1 \leq \mu$ and there exists $\varphi = \frac{d\mu_1}{d\mu}$, $\varphi \geq 0$ and $\mu_1(B) = \int_B \varphi \, d\mu \quad \forall B \in \mathcal{B}$.

Let $E = \{x : \varphi(x) < 1\}$ Now using the easy Set theory fact that $A = A \cap B \amalg A - B$ twice and invariance of the measure μ_1

$$\int_{E\wedge f^{-1}E} \varphi \, d\mu + \int_{E-f^{-1}E} \varphi \, d\mu = \int_E \varphi \, d\mu = \mu_1(E)$$

$$= \mu_1(f^{-1}E) = \int_{f^{-1}E} \varphi \, d\mu = \int_{f^{-1}E\wedge E} \varphi \, d\mu + \int_{f^{-1}E-E} \varphi \, d\mu$$

and so $\displaystyle\int_{E-f^{-1}E} \varphi \, d\mu = \int_{f^{-1}E-E} \varphi \, d\mu$

Now $\varphi < 1$ on $E - f^{-1}E$ and $\varphi \geq 1$ on $f^{-1}E - E \subseteq E^c$ using the definition of E. Also $\mu(f^{-1}E - E)$

$$= \mu(f^{-1}E) - \mu(f^{-1}E \wedge E) = \mu(E) - \mu(f^{-1}E \wedge E)$$

$= \mu(E - f^{-1}E)$ using invariance of measure and $A = A \wedge B \amalg A - B$ again

Thus $\int_{E-f^{-1}E} \varphi\, d\mu = \int_{f^{-1}E-E} \varphi\, d\mu$ is

possible only if $\mu(f^{-1}E-E)=0=\mu(E-f^{-1}E)$

Now $f^{-1}E \Delta E = f^{-1}E-E \perp\!\!\!\perp E-f^{-1}E$ and so

$\mu(f^{-1}E\Delta E)=0$. But f is ergodic so

either $\mu(E)=1$ or $\mu(E)=0$. Assume $\mu(E)=1$

then since $\varphi<1$ on E. $1=\mu_1(X)=\int_E \varphi\, d\mu < \mu(E)=1$

an contradiction, Thus $\mu(E)=0$.

Letting $F=\{x : \varphi(x)>1\}$ a similar argument

yields $\mu(F)=0$. Thus $\mu\{x : \varphi(x)=1\}=1$ i.e.

$\varphi\equiv 1$ a.e. Thus $\mu_1(B)=\int_B d\mu=\mu(B)$ and

So $\mu_1=\mu$ and $p=1$. $\blacksquare$

__COMMENT__: $M(X, f)$ is a compact, convex set but it is in infinite dimensions so strange things can happen. For example, the next theorem says that for (Σ_2, σ) the extreme points are dense, so, in particular, $M(X, f)$ has no interior since every open set contains an extreme point.

__Theorem__ (Parthasarathy). In $M(\Sigma_2, \sigma)$ the collection of invariant measures on periodic points is dense in the weak*-topology

__NOTE__: The invariant measure on a periodic point is ergodic and thus is an extreme point of $M(\Sigma_2, \sigma)$

<u>Idea of the proof:</u> Let $B_1, B_2, \ldots$ be an enumeration of the blocks of 0's and 1's in an order of non decreasing lengths. eg

$0, 1, 00, 01, 10, 11, 000, \ldots$ \mu(B) is short for \mu([B]) below

Recall that there is a metric on $M(\Sigma_2)$

$$d(\mu_1, \mu_2) = \sum_{n=1}^{\infty} \frac{|\mu_1(B_n) - \mu_2(B_n)|}{2^n}$$

Given, $\varepsilon > 0$ and a measure $\mu \in M(\Sigma_2, \sigma)$ we find a periodic orbit measure ν with $d(\mu, \nu) < \varepsilon$

Now $\exists N$ with $\sum_{n=N}^{\infty} \frac{1}{2^n} < \varepsilon/2$ so we just need $\mu_{\#}(B_n)$ and $\nu(B_n)$ close for $n < N$.

If $\mu([0]) = p_0$ $\mu([1]) = p_1 = 1 - p_0$ start with a periodic orbit whose proportion of 0's is closest to p_0 then $|\mu[0] - \nu[0]|$ and $|\mu[1] - \nu[1]|$ are both small. Now adjust the periodic orbit so its proportion of 00, 01, 10, 11 are close to $\mu[00]$, $\mu[01]$, $\mu([1,0])$, $\mu([11])$. Continue up to N.

It takes some work to do this carefully and one has to use the fact that μ is additive so $\mu[0] = \mu[00] + \mu[01]$, etc. $\blacksquare$

<u>Regular points</u>: A point $x \in X$ is regular for f, μ if

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

for <u>all</u> $\alpha \in C(X, \mathbb{R})$

The regular set $\mathcal{R}(\mu, f)$ is the collection of all regular points

<u>Theorem</u> (Oxtoby) With hypothesis as above, μ ergodic under f, $\mu(\mathcal{R}(\mu, f)) = 1$ so a.e. point is regular.

<u>Proof</u> Recall that $C(X, \mathbb{R})$ is separable and so there is a countable, dense set $\{\alpha_k\}$ $k = 1, .., \infty$.

By the ergodic theorem $\exists R_k \subseteq \mathcal{B}$ so that

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha_k \circ f^i(x) \to \int \alpha_k \, d\mu$$

for all $x \in R_k$ and $\mu(R_k) = 1$.

Thus $x \in R = \bigcap_{k=1}^{\infty} R_k$ has

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha_k \circ f^i(x) \to \int \alpha_k \, d\mu$$

for all k and $\mu(R) = 1$.

To finish, we need to approximate an arbitrary α by α_k. So we need this

if $\alpha_j \to \alpha$ in $C(X, \mathbb{R})$

and $\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha_j \circ f^i(x) \to \int \alpha_j \, d\mu$

for all j $\Rightarrow$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

which follows since $\alpha_j \to \alpha$ means $\alpha_j \to \alpha$ uniformly and f is continuous $\blacksquare$

Recall that μ and ν are mutually singular (both Borel) if there exists $B \in \mathcal{B}$ with $\mu(B)=0$ and $\nu(B)=1$ (and so $\mu(B^c)=1$ and $\nu(B^c)=0$). Written $\mu \perp \nu$

<u>CORR</u>: If μ and ν are ergodic then $\mu \perp \nu$. (assuming $\mu \neq \nu$)

<u>Proof</u> We claim that $R(\mu, f) \wedge R(\nu, f) = \phi$.

Since $\mu(R(\mu, f)) = 1$ and $\nu(R(\nu, f)) = 1$ and $R(\nu, f) \subseteq R(\mu, f)^c$ so $\mu(R(\nu, f)) = 0$, This will finish it.

Assume then $\exists\, x \in R(\mu, f) \wedge R(\nu, f)$, then

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu \qquad \text{for all } \alpha \in C(X, \mathbb{R})$$

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\nu$$

Thus $\int \alpha \, d\mu = \int \alpha \, d\nu$, $\forall \alpha \in C(X, \mathbb{R})$, which we showed above implies $\mu = \nu$. ∎

<u>Example</u>: let $\vec{p} = (p_1, p_2)$ $p_1 + p_2 = 1$, $p_i > 0$ and let $\mu_{\vec{p}}$ be the Bernoulli measure on Σ_2. We showed $\mu_{\vec{p}}$ is ergodic under σ, thus

$\mu_{\vec{p}}(\mathcal{R}(\mu_{\vec{p}}, \sigma)) = 1$ for all $\vec{p}$. In addition, each $\mu_{\vec{p}}$ is positive on open sets so $spt(\mu_{\vec{p}}) = \Sigma_2$ for all $\vec{p}$. And by the COR, with $\vec{p} \neq \vec{p}'$, $\mathcal{R}(\mu_{\vec{p}}, \sigma) \cap \mathcal{R}(\mu_{\vec{p}'}, \sigma) = \emptyset$.

So there are uncountably many disjoint regular sets, all dense in Σ_2, each regular set is full measure for its own $\mu_{\vec{p}}$ and zero measure for all the other $\mu_{\vec{p}}$.