ET19

1

Continuous Ergodic Theory, cont

- $f : X \circlearrowleft$ cpt metric, f continuous $\mu \in M(X, f)$
- A point x is regular for μ and f if

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int_X \alpha \, d\mu, \quad \underline{\forall \alpha \in C(X, \mathbb{R})}$$

- $R(f, \mu)$ is the collection of regular points
- If μ is ergodic under f, then $\mu(R(f, \mu)) = 1$.

Recall we found invariant measures as weak limit points of $\{\mu_n\}$ where

$$\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} f_*^i(\delta_x)$$

- When μ is ergodic, the next theorem says that for a.e. point x, this weak* asymptotic average along the orbit of x actually converges to μ.

- We need a result from HW4: μ_n as above

$\mu_n \to \mu$ in the weak* topology

$$\Leftrightarrow \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

- **Theorem**: If μ is ergodic and $x \in R(f, \mu)$ (recall $\mu(R(f,\mu)) = 1$) then

$$\frac{1}{n} \sum_{i=0}^{n-1} f^i_* \delta_x \to \mu.$$

- Proof: Apply the HW to the definition of regular

- Notice that regular is stronger than just the Ergodic Thm.
- We examine the convergence of time averages to space averages seen so far.

- __Pointwise ergodic theorem__: $f:(X,\mathcal{B},\mu)\circlearrowleft$ is mpt and ergodic. Given $\alpha \in L^1(\mu)$ $\exists$ full measure set Y so that $x \in Y$ implies

$$\frac{1}{n}\sum_{i=0}^{n-1} \alpha \circ f^i(x) \longrightarrow \int \alpha \, d\mu$$

- __Regularity__ $f:(X,\mathcal{B},\mu)$ is mpt ergodic and continuous
$\exists$ full measure set Y such that for all $\alpha \in C(X,\mathbb{R})$

$x \in Y$ $\quad \frac{1}{n}\sum_{i=0}^{n-1} \alpha \circ f^i(x) \longrightarrow \int \alpha \, d\mu$

__Notice__: one full measure set works for all α
but α now continuous not just L^1

Condition ?: $f : (X, \mathcal{B}, \mu)$ is mpt, ergodic and continuous and $\boxed{?}$ then for all $x \in X$ and all $\alpha \in C(X, \mathbb{R})$

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

Remark: So time averages = space averages for every point in X OR ~~$R(X)$~~ $R(f, \mu) = X$

DEF: $f : X \to X$, X cpt metric, f cont. is called uniquely ergodic if it has exactly on invariant Borel probability measure.

Consequence: $M(X, f)$ is a singleton, $\{\mu\}$, which is an extreme point and thus ergodic

Example (to be shown): $R_\alpha : S^1 \circlearrowleft$ $\theta \mapsto \theta + \alpha$, $\alpha \notin \mathbb{Q}$ is uniquely ergodic, & $M(S^1, R_\alpha) = \{\text{Lebesgue}\}$

$\sigma : \Sigma_2 \circlearrowleft$ is not uniquely ergodic, $M(\Sigma_2, \sigma)$ contains Markov measures, Bernoulli measures, periodic orbit measures, ...

Theorem $f : X \to X$ cont on compact metric TFAE

(i) $\forall \alpha \in C(X, \mathbb{R})$, $\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i$ converges to a constant uniformly

(ii) $\forall \alpha \in C(X, \mathbb{R})$, $\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i$ converges to a constant pointwise

(iii) $\exists \mu \in M(X, f)$ such that $\forall \alpha \in C(X, \mathbb{R})$, $\forall x \in X$

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

(iv) f is uniquely ergodic

PROOF: (i) $\Rightarrow$ (ii) is trivial

(ii) $\Rightarrow$ (iii) let $L(\alpha) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \alpha\circ f^i$

By hypothesis, $L(\alpha)$ exists for all α so $L: C(X, \mathbb{R}) \to \mathbb{R}$.

It is easy to see $L(a\alpha + b\beta) = aL(\alpha) + bL(\beta)$ so L is linear.

Since $\alpha \in C(X, \mathbb{R})$ and X is compact, α is bounded

(recall $\|\alpha\|_0 = \max\{\alpha(x) : x \in X\}$) and so by

the triangle inequality $\left|\frac{1}{n}\sum_{i=0}^{n-1} \alpha\circ f^i\right| \leq \frac{1}{n}\sum \|\alpha\|_0 = \|\alpha\|_0$

This implies that $|L(\alpha)| \leq \|\alpha\|_0$ and so L is a bounded, linear functional on $C(X, \mathbb{R})$

Thus by the Riesz Representation Theorem $\exists\, \mu \in M(X)$ with $L(\alpha) = \int \alpha\, d\mu$. Now note $L(\alpha\circ f) = \lim_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} \alpha\circ f^{i+1} = L(\alpha)$ by an argument we have used several times

Since $L(\alpha) = L(\alpha \circ f)$ we have

$$\int \alpha \, d\mu = \int \alpha \circ f \, d\mu \quad \forall \alpha \in C(X, \mathbb{R})$$

We showed Ex 1b pg 11 that This implies $\mu \in M(X, f)$

and by construction $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) = \int \alpha \, d\mu; \ \forall \alpha \in C(X, \mathbb{R})$

(iii) $\Rightarrow$ (iv) By hypothesis $\forall \alpha \in C(X, \mathbb{R}), \forall x \in X,$

$$\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i(x) \to \int \alpha \, d\mu$$

Now assume $\nu \in M(X, f)$. Since α is bounded

and so $\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i$ is bounded by the same bound

$$\int \left(\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i \right) d\nu \to \int \left(\int \alpha \, d\mu \right) d\nu$$

by the bounded convergence theorem

but ν is f-invariant and we showed in a HW 8

$$\int \left(\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i \right) d\nu = \int \alpha \, d\nu$$

and $\int \left(\int \alpha \, d\mu \right) d\nu = \int \alpha \, d\mu$ since $\int \alpha \, d\mu$ is a constant and $\nu(X) = 1$

Thus $\int \alpha \, d\nu = \int \alpha \, d\mu$ for all $\alpha \in C(X, \mathbb{R})$

and we showed previously that this implies (Lemma page 3, ET 16) $\nu = \mu$, so there exists a unique $\mu \in M(X, f)$.

$(iv) \Rightarrow (i)$ We prove the contrapositive

First recall uniform convergence $g_n \rightrightarrows g$

$\forall \varepsilon > 0 \ \exists N \ \forall n \geq N \ \forall x$

$$|g_n(x) - g(x)| < \varepsilon$$

To negate this $\exists \varepsilon > 0 \ \forall N \ \exists n \geq N \ \exists x$

$$|g_n(x) - g(x)| \geq \varepsilon$$

Now note that if $\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i$ converges uniformly to a constant, By the ergodic theorem, that constant must be $\int \alpha \, d\mu$ where $\{\mu\} = M(X, f)$

We must negate (i): $\forall \alpha \in C(X, \mathbb{R})$ $\frac{1}{n} \sum_{i=0}^{n-1} \alpha \circ f^i$ converges uniformly to a constant

Thus $\exists \beta \in C(X, \mathbb{R})$, $\exists \varepsilon > 0$ $\forall N$ $\exists n > N$

and $\exists x_n \in X$ with

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} \beta \circ f^i(x_n) - \int \beta \, d\mu \right| \geq \varepsilon$$

Now let $\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \delta_{f^i(x_n)}$ and so

$$\int \beta \, d\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \beta \circ f^i(x_n)$$ as noted above (HW)

Thus $\left| \int \beta \, d\mu_n - \int \beta \, d\mu \right| \geq \varepsilon$

Now using the compactness of $M(X)$

find $\mu_\infty \in M(X)$ and $n_i \to \infty$ with

$\mu_{n_i} \to \mu_\infty$ weak*

Using an argument just like that of Theorem on page 3 of ET17, we have $\mu_{\infty} \in M(X, f)$ i.e. it is an invariant measure.

Further since $\mu_{n_i} \to \mu_{\infty}$ weakly, $\int \beta \, d\mu_{n_i} \to \int \beta \, d\mu_{\infty}$ and so

$$\left| \int \beta \, d\mu_{\infty} - \int \beta \, d\mu \right| \geq \varepsilon$$

Thus $\mu_{\infty} \neq \mu$ and $\mu_{\infty} \in M(X, f)$, and so f is not uniquely ergodic.