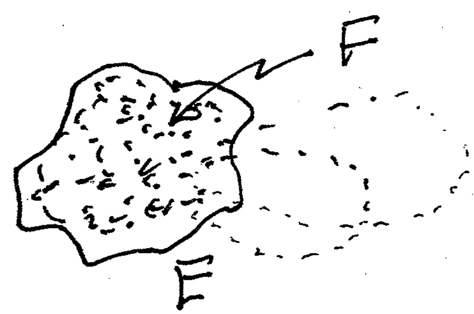
Poincaré Recursion Cont.

First the version in just a measure space

<u>Theorem</u> Let $f: X \to X$ be a measure preserving transformation of a finite measure space $(X, \mathcal{B}, \mu)$. Assume $E \in \mathcal{B}$ has $\mu(E) > 0$. Then a.e. point of E returns to it infinitely often in forward time under f or in more detail, $\exists F \in \mathcal{B}$, $F \subseteq E$ and $\mu(F) = \mu(E)$ such that $\forall x \in F \ \exists \ n_i \to \infty$ so that $f^{n_i}(x) \in F \subseteq E$ for all i.



<u>Proof</u> For each integer $N \geq 0$ let

$$E_N = \bigcup_{n=N}^{\infty} f^{-n}(E) \text{. Thus } x \in E_N \Rightarrow$$

$x \in f^{-n}(E)$ for some n so $f^n(x) \in E$.

So E_N is all x which enter E via f^n for some $n \geq N$.

Thus $\bigcap_{N=0}^{\infty} E_N$ is precisely those x which return to E infinitely often in forward time

The set we are interested in is

$$F = E \cap \left(\bigcap_{N=0}^{\infty} E_N \right)$$

which is all points in E which return to E infinitely often in forward time.

(3)

So if $x \in F$ $\exists\ 0 < n_1 < n_2 < \ldots$ with $f^{n_i}(x) \in E$ for all i. But in fact since $f^{n_j}(x) = f^{n_j - n_i}(f^{n_i}(x)) \in E$ we have that $j \geq i$ $f^{n_j - n_i}(f^{n_i}(x)) \in E$ for the ∞ sequence in j, $n_j - n_i$, and thus $f^{n_i}(x) \in F$ by its definition. Thus when $x \in F$ we have that $f^n(x)$ returns to $\underline{\underline{F}}$ infinitely often in forward time.

To finish, we need to show that F is full measure in E, i.e. $\mu(F) = \mu(E)$.

From the definition, $f^{-1}(E_N) = E_{N+1}$ and since f preserves μ, $\mu(E_N) = \mu(E_{N+1})$ then by induction $\mu(E_N) = \mu(E_0)$ for all N.

Also from the definition, $E_0 \supset E_1 \supset E_2 \supset \ldots$

and all E_n have the same measure and so

$$\mu\left(\bigcap_{n=0}^{\infty} E_n\right) = \mu(E_0) \quad \text{(HW Lemma).}$$

Now since $f^{-0}(E) = E$, $E \subset E_0 = \bigcup_{n=0}^{\infty} f^{-n}(E)$

Now another general fact: if $B \subseteq C$ and $\mu(B) = \mu(C)$

then $\mu(A \cap B) = \mu(A \cap C)$

Thus since $\bigcap_{n=0}^{\infty} E_n \subseteq E_0$ and $\mu\left(\bigcap_{n=0}^{\infty} E_n\right) = \mu(E_0)$

and $F = E \cap \bigcap_{n=0}^{\infty} E_n$ then by the general fact

$\mu(F) = \mu(E \cap E_0)$. But we showed above

that $E \subset E_0$ and so $\mu(F) = \mu(E \cap E_0) = \mu(E)$

as required. $\blacksquare$

For the topological version we need the notion of a base for a topology: So Assume X has a topology (which you can think of as coming from a metric)

A collection $\mathcal{C}$ is a base for the topology if every open set U in X can be written as

$$U = \bigcup_{\alpha\in A} C_\alpha \quad \text{with } C_\alpha \subseteq \mathcal{C}.$$

Notice that this union could be uncountable.

MAIN Example: Let X be a compact metric space. For each n, consider the cover $B_{1/n}(x)$ for all $x \in X$. By compactness, it has a finite subcover say $B_{1/n}(x_{i,n})$ for $i = 1, \dots, N(n)$ for some finite N.

then $\bigcup_{n=1}^{\infty} \bigcup_{i=1}^{N(n)} B_{1/n}(x_{i,n})$ is a base for

the topology.

Important feature: This base is countable

- If $(X, \mathcal{T})$ has a countable base it is called <u>2nd countable</u>
- If $(X, \mathcal{T})$ has a countable dense set, it is called <u>separable</u>
- In metric spaces 2nd countable $\Leftrightarrow$ separable.

<u>Caution</u>: What we gave is a consequence of being a base

Formall $\mathcal{C}$ is a collection of open sets in X

Such that (a) $\bigcup C_\alpha : \alpha \in \mathcal{C}$ covers X

(b) $\forall C_1, C_2 \in \mathcal{C}$ if $x \in C_1 \wedge C_2$

$\Rightarrow \exists C_3$ with $x \in C_3$

and $C_3 \subseteq C_1 \wedge C_2$.

A base plays the role like a basis in Linear Algebra - it suffices often to just check a property on the basis and then it holds for the whole space.

Examples - Assume X has a countable basis $\{U_i\}$ for its topology (eg X is compact metric)

- $x_n \to x_0 \iff \forall U_i$ with $x_0 \in U_i$ $\exists N$ so that $n \geq N \Rightarrow x_n \in U_i$

- $Z \subseteq X$ is dense $\iff \forall i,\ Z \cap U_i \neq \emptyset$

- Now $f: X \to X$: A point x is recurrent $\iff \forall U_i$ with $x \in U_i$ $\exists n$ with $f^n(x) \in U_i$

Poincaré recurrence: continuous version

Theorem: Let X be a compact metric space, $f: X \to X$ is continuous and preserves a Borel probability measure, then almost every point is recurrent i.e. $\exists F \in \mathcal{B}$ with $\mu(F) = 1$ and $x \in F \Rightarrow x$ is recurrent

Proof: Let $\{U_i\}_{i \in \mathbb{N}}$ be a countaske base. By the measure theoretic Poincaré recurrest, $\forall i$ $\exists F_i \subseteq U_i$ with $\mu(F_i) = \mu(U_i)$ and $x \in F_i \Rightarrow \exists n$ with $f^n(x) \in F_i \subseteq U_i$. Now $X = F_i \amalg U_i - F_i \amalg U_i^c$ (where $A \amalg B$ means $A \cup B$ and $A \cap B = \emptyset$). since $\mu(U_i - F_i) = 0$, $\mu(F_i \amalg U_i^c) = 1$. Let

$$W_i = F_i \amalg U_i^c \text{ and so } \mu(W_i) = 1$$

Thus $F := \bigcap_{i=0}^{\infty} W_i$ also has $\mu(F) = 1$

But note that if $x \in W_i$ then either $x \in F_i$ or $x \in U_i^c$. But $\bigcup U_i = X$ so $x \in F$ means that it returns to every F_i it belongs to ~~or~~ thus it returns to every base element U_i it returns to, so x is recurrent ∎

DEF (9) If μ is a Borel measure in the metric space X the support of μ, written, $\operatorname{supp}(\mu)$ is the complement of the union of all the open sets of zero measure

$$\operatorname{supp}(\mu) := X - \bigcup \{U : U \text{ is open and } \mu(U) = 0\}$$

So $\operatorname{supp}(\mu)$ is closed.

HW: $\operatorname{supp}(\mu) = \bigcap \{C : C \text{ is closed and } \mu(C) = 1\}$

(b) The recurrent set of $f: X \to X$ (still in Topological situation) is

$$\mathcal{R}(f) = \{x \in X : x \text{ is recurrent}\}$$

CORR: under the hypothesis of the continuous Poincaré Recurrence, $\operatorname{supp}(\mu) \subseteq \overline{\mathcal{R}(f)}$ (over bar means closure)

Example: Let $S^1 = \mathbb{R}/\mathbb{Z} = [0,1]/\sim$ be the circle

$R_\alpha : S^1 \supseteq$ defined by $R_\alpha(\theta) = \theta + \alpha$

(a) $\alpha = p/q$ in lowest terms. Every point is periodic of period q. $\{x, R(x), \dots, R^{q-1}(x)\}$ is called a periodic orbit. It is recurrent. Define a measure μ on it $\mu(R^i(x)) = 1/q$ for all i.

Then μ in $R_{p/q}$-invariant, probability and x is recurrent

(a) continued: Let m be the measure induced on S^1 by Lebesque on $[0,1]$ or just induced by arclength. Then m is also $R_{p/q}$ invariant and a.e. point (every point) is recurrent. In general a given Topological system can have many invariant measures.

(b) $\alpha \notin \mathbb{Q}$, irrational rotation. Now m is still invariant, every point is recurrent. In fact (to be proved) m is the only R_α invariant Borel probability measure.