Unique ergodicity - examples

- Many examples of uniquely ergodic transformations come from translations on topological groups

- Assume G is a group which we write multiplicatively and also a topological space

- G is a topological group if the two operations

$$(x,y) \longmapsto xy \qquad G \times G \to G$$

$$x \longmapsto x^{-1} \qquad G \to G$$

are continuous ($G \times G$) is given the product topology.

- Some examples: $(\mathbb{R}^n, +)$, $(\mathbb{R} - \{0\}, \cdot)$, $\mathbb{T}^n = S^1 \times \cdots \times S^1$, $GL(n,\mathbb{R})$ invertible $n \times n$ matrices under multiplication with topology $M \subseteq \mathbb{R}^{n^2}$.

- If G is a topological group a measure μ on G is called left invariant if for $\forall g \in G$ and all $B \in \mathcal{B} =$ Borel σ-algebra $\mu(gB) = \mu(B)$ where $gB = \{gb : b \in B\}$.

- <u>Haar measure</u>: Let G be a <u>compact, metric</u> topological group then there exists a unique left invariant Borel probability measure μ.

- Note that μ is also right invariant, $\mu(Bg) = \mu(B)$.

<u>proof</u>: Fix $x \in G$ and define $\nu(B) = \mu(Bx)$.

Then using left invariance μ,

$\nu(gB) = \mu(gBx) = \mu(Bx) = \nu(B)$.

ν is left invariant, so $\nu = \mu$.

Thus $\mu(Bx) = \mu(B)$, for all x.

This is true for all x, so

Example: Lebesgue measure on S^1 is Haar measure and on $\mathbb{T}^n = S^1 \times \ldots \times S^1$ the n-torus, Haar is the product of Lebesgue measure on each circle

Alternative Definition: Just like the RESULT for an invariant measure for a transformation, μ is Haar measure on G

$$\Leftrightarrow \int_G \alpha(gx)\, d\mu(x) = \int_G \alpha(x)\, d\mu(x) \quad \forall \alpha \in C(X, \mathbb{R}), \ \forall g \in G.$$

DEFINITION: Fix $g \in G$, the translation transformation coming from g is $f_g : G \to G$ via

$$f_g(x) = gx \quad \text{(writing multiplicatively)}$$

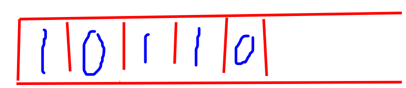
Remark f_g is a homeomorphism with $f_g^{-1}(x) = g^{-1}x$

Examples: $R_\alpha : S^1 \to S^1$ w/ $R_\alpha(\theta) = \theta + \alpha$ is a translation as is $f : \mathbb{T}^n \to \mathbb{T}^n$ via

$$(\theta_1, \dots, \theta_n) \to (\theta_1 + \alpha_1, \dots, \theta_n + \alpha_n) \mod 1 \text{ in all components.}$$

Example: The adding machine (informally)

Let Σ_2^+ have the addition given by addition with carrying



$$\begin{array}{r} .01100\dots0 \\ +\ .01000\quad 0 \\ \hline .0001000\dots \end{array}$$

is a compact, topological group

$f : \Sigma_2 \circlearrowleft$ defined by $\underline{s} + (.1000\dots)$ is called the adding machine or binary odometer.

- Some translations on compact groups are uniquely ergodic some are not. For example, $R_{p/q} : S^1_2$ has uncountably many invariant measures on for each periodic point.

- The crucial hypothesis is minimality, a topological condition. We just define it for homeomorphisms: There is also a definition for non-injective $f : X \to X$

- **Def**: A homeomorphism $f : X \to X$ (of a cpt metric) is minimal if $\forall x \in X$, $\overline{o(x,f)} = X$ ie the orbit of every point is dense, $o(x,f) = \{\ldots, f^{-1}(x), x, f(x), \ldots\}$

- Minimality has similarities to ergodicity but in the topological category.

<u>Theorem</u> $h : X \circlearrowleft$ is a homeomorphism of a compact metric space. TFAE

(i) h is minimal

(ii) The only cpt invariant sets $E \subset X$ ($h(E) = E$) are $E = \emptyset$ or $E = X$

(iii) If $U \neq \emptyset$ and is open $\bigcup_{n \in \mathbb{Z}} h^n U = X$

<u>Proof</u>: (i) $\Rightarrow$ (ii) Say E is cpt, $E \neq \emptyset$ and $h(E) = E$ then for $x \in E$, $O(x, h) \subseteq E$ and so $X = \overline{O(x,h)} \subseteq \overline{E} = E$

(ii) $\Rightarrow$ (iii) If $U \neq \emptyset$ and is open $E = X - \bigcup_{n \in \mathbb{Z}} h^n(U)$ is cpt and $h(E) = E$ but $E \neq X$ so $\emptyset = X - \bigcup_{n \in \mathbb{Z}} h^n(U)$

so $\bigcup_{n \in \mathbb{Z}} h^n(U) = X$

(iii) $\Rightarrow$ (i) let $x \in X$ and open $U \neq \emptyset$ be arbitrary. By hypotheses $x \in \bigcup_{n \in \mathbb{Z}} h^n U$ and so $\exists N$ so that $x \in h^N(U)$ or $h^{-N}(x) \cap U \neq \emptyset$ Thus $O(x, h)$ intersects every non-empty open set so $\overline{O(x,h)} = X$ $\blacksquare$

GP

We need 2 facts before the result characterizing unique ergodicity for translations on compact groups

(1) when G is compact, metric Haar measure is positive on open sets

Proof Certainly $G = \bigcup_{g \in G} gU$, This yields an open cover of G. By compactness, $G = g_1 U \cup \ldots \cup g_n U$ for some finite set $\{g_1, \ldots, g_n\}$ but $\mu(g_i U) = \mu(U)$ and if $\mu(U) = 0 \Rightarrow \mu(G) = 0$, a contradiction

(2) $h: X \circlearrowleft$ is a homeomorphism of a compact metric (not just a group translation) and h is uniquely ergodic with, $M(X, h) = \{\mu\}$ TFAE

(i) h is minimal

(ii) μ is positive on open sets

Proof: (i) $\Rightarrow$ (ii) If h is minimal, by the last Theorem iii when U is open, $X = \bigcup_{n \in \mathbb{Z}} h^n(U)$ if $\mu(U) = 0 \Rightarrow \mu(h^n(U)) = 0$ and so $\mu(X) = 0$, a contradiction.

(ii) $\Rightarrow$ (i) Now conversely, assume μ is positive on open sets and say h is not minimal. Thus by previous Theorem (ii), $\exists$ cpt set K with $h(K)=K$ and $K \neq \phi, \mathfrak{X}$. By the theorem on page 3 of ETII, $h|_K$ has an invariant measure μ_K. Define ν on $\mathfrak{X}$ by $\nu(B) = \mu_K(K \cap B)$ for all $B \in \mathcal{B}$. Then $\nu \in M(\mathfrak{X}, h)$ and $\nu \neq \mu$ since $\nu(K^c) = 0$ but $\mu(K^c) > 0$ since K^c is open and nonempty. Thus h has two invariant measures, contradicting unique ergodicity $\blacksquare$

Since we showed for any $\mu \in M(\mathfrak{X}, h)$, $h(\mathrm{spt}(\mu)) = \mathrm{spt}\,\mu$ and μ is post. on open sets in $\mathrm{spt}(\mu)$ and $\mathrm{spt}(\mu)$ is compact,

<u>COR</u>: If $h: \mathfrak{X} \circlearrowleft$ is a uniquely ergodic homeomorphism of a compact metric space then $h|_{\mathrm{spt}(\mu)}$ is a minimal homeomorphism of $\mathrm{spt}(\mu)$ i.e. $\mathrm{spt}(\mu)$ is a minimal set for h.

Now back to unique ergodicity

Theorem: Let G be a compact metric group and $g \in G$. $h_g : G \to G$ is translation $h_g(x) = gx$ (we write multiplicative)

TFAE

(i) h_g is minimal

(ii) $\{g^n : n \in \mathbb{Z}\}$ is dense in G (This is $o(g, h_g)$)

(iii) h_g is uniquely ergodic and the unique invariant measure is Haar measure

Proof (i) $\Rightarrow$ (ii) If h_g is minimal $\Rightarrow$ $o(g, h_g)$ is dense.

(ii) $\Rightarrow$ (i) Assume $\{g^n : n \in \mathbb{Z}\}$ is dense. Pick $x \in G$. We need $o(x, h_g) = G$. Now pick any $y \in G$, $\exists\, n_i$ so that $g^{n_i} \to yx^{-1}$ and so $g^{n_i} x \to y$ or

$h_g^{n_i}(x) \to y$ as needed

(iii) $\Rightarrow$ (i) By the previous general theorem,

$\circledast$ since Haar measure is positive on open sets

unique ergodicity $\Rightarrow$ minimality.

(ii) $\Rightarrow$ (iii) Say $\mu \in M(G, h_g)$. Then by change of variables

$$\int \alpha(g^n x)\, d\mu(x) = \int \alpha(h_g^n(x))\, d\mu(x) = \int \alpha(x)\, d\mu(x)$$

$\forall \alpha \in C(X, \mathbb{R})$, $\forall n \in \mathbb{Z}$.

Given $y \in G$, $\exists\, g^{n_i} \to y$ so by dominated convergence

$$\lim_{i\to\infty} \int \alpha(g^{n_i} x)\, d\mu(x) = \int \alpha(yx)\, d\mu(x)$$

and so $\int \alpha(yx)\, d\mu(x) = \int \alpha(x)\, d\mu(x)$.

This says as noted above, μ is invariant under left translation by y, $\forall y \in G$, and so μ is Haar measure. $\boxtimes$

Examples

(1) $R_\alpha : S^1 \to S^1$ with $\alpha \notin \mathbb{Q}$. It follows from number theory (or a direct proof) that $\{\alpha n\} \bmod 1$ is dense in S^1 over $n \in \mathbb{Z}$ $\Rightarrow$ R_α is uniquely ergodic

(2) If $\vec{\alpha} \in \mathbb{R}^2$ is rationally independent we showed $(\theta_1, \theta_2) \mapsto (\theta_1 + \alpha_1, \theta_2 + \alpha_2) \bmod 1$ is ergodic on $\mathbb{T}^2$. It turns out to be minimal also and so it is uniquely ergodic

(3) The adding machine: Recall two sequences are close if they agree on a big initial chunk.
$h : \Sigma_2^+ \circlearrowleft$ $\quad h(\underline{s}) = \underline{s} + .10000\ldots$ is Translation by $.1000\ldots$. Given any initial block $.b_0 b_1 \ldots b_k$ let $m = b_0 + 2b_1 + 4b_2 + \ldots + 2^k b_k \Rightarrow m \cdot (.1000\ldots) = .b_0 b_1 \ldots b_k 000$ so the orbit of $.1000$ is dense and so h is minimal and uniquely ergodic.

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(4) For those in last semesters course:
Build a subshift using the fixed point of a primitive substitution. It is uniquely ergodic.

Note this is not translation on a compact group