ET21

Isomorphism and conjugacy

- Most areas of mathematics have a notion of when two systems are equivalent.
- Isomorphic groups, homeomorphic topological spaces, isometric metric spaces.
- In Topological dynamics the notion is topological conjugacy

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha"'] & X \arrow[d, "\alpha"] \\ Y \arrow[r, "g"] & Y \end{tikzcd}$$

α is a homeomorphism, (X, f) is topologically conjugate to (Y, g). They have all the same dynamics in topological sense

· There are two notions of equivalence in ergodic theory arising from two notions of equivalence in measure theory: isomorphism and conjugacy.

· We need a bit of measure theory first

<u>DEF</u>: (1) A probability measure space is called <u>complete</u> if every subset of a measure zero set is measurable.

(2) The <u>completion</u> of $(X, \mathcal{B}, \mu)$ is $(X, \mathcal{B}_\mu, \mu)$ where $\mathcal{B}_\mu$ is the smallest sigma algebra that contains both $\mathcal{B}$ and all the subsets of μ-measure zero sets.

<u>Example</u> If $([0,1], \mathcal{B}, \mu)$ is Lebesgue measure on the Borels, $\mathcal{B}$'s completion is often called the Lebesgue measurable sets

DEF: the probability spaces $(X_1, \mathcal{B}_1, \mu_1)$ and $(X_2, \mathcal{B}_2, \mu_2)$ are <u>isomorphic</u> if $\exists\, M_i \in \mathcal{B}_i$, $\mu_i(M_i) = 1$ $i = 1, 2$ and an bi-measurable bijection bi-measure preserving $\phi : M_1 \to M_2$

<u>RK</u> M_i has the σ-algebra $M_i \cap \mathcal{B}_2$ and measure μ_i restricted to M_i.

<u>NOTE</u> So the spaces are the same measure-theoretically after we neglect a set of measure zero from both spaces.

<u>Example</u>: Let $D \subseteq [0,1]$ be all points of the form $\frac{k}{2^n}$ $0 \le k \le 2^n$. $D_2 \subseteq \Sigma_2^+$ be all sequences that do <u>not</u> end with 0^∞ or 1^∞ and $\psi : \Sigma_2^+ \to [0,1]$ via $\psi(\underline{s}) = \frac{s_0}{2} + \frac{s_1}{2^2} + \frac{s_2}{2^3} + \cdots$

Give Σ_2^+ the $(1/2, 1/2)$-Bernoulli measure and $[0,1]$ Lebesque then $\psi : \Sigma_2^+ - D_2 \to [0,1] - D$

and D_2 and D both have measure zero.

is bijective and ~~by~~ bi-measure preserving and so the probability measure spaces are isomorphic

- An *atom* for a probability measure space $(X, \mathcal{B}, \mu)$ is a point x with $\mu(\{x\}) > 0$. The space is called *nonatomic* if it has no atoms

- It turns out that with topological hypothesis probability spaces are isomorphic to Lebesque measure

- *Theorem* If X is separable metric, $\mathcal{B}$ its Borels and μ a nonatomic probality measure $\Rightarrow$ $(X, \mathcal{B}, \mu)$ is isomorphic to $[0,1]$ with the Borels and Lebesque measure. The completion of $(X, \mathcal{B}, \mu)$ is isomorphic to $[0,1]$ with Lebesque measurable sets (i.e. the completion of the Borels) and Lebesque measure.

- A space isomorphic to Lebesque is called Lebesque space.

· Another more mathematically elegant way to deal with measure zero sets (that is harder to work with in examples) is to use measure algebras

· Let $(X, \mathcal{B}, \mu)$ be a probablity measure space. As in our homework, define an equivalence relation for $A, B \in \mathcal{B}$ as $A \sim B \Leftrightarrow \mu(A \Delta B) = 0$. Let $\tilde{\mathcal{B}}$ be the collection of equivalence classes. Define

$$[A]^c = [A^c], \quad [A] \wedge [B] = [A \wedge B]$$

$$[A] \cup [B] = [A \cup B]$$

These are all well defined

This makes $\tilde{\mathcal{B}}$ into a Boolean σ-algebra i.e. all the σ-algebra properties hold. Finally,

$\tilde{\mu}([B]) = \mu(B)$ is also well-defined.

$(\tilde{\mathcal{B}}, \tilde{\mu})$ is the associated <u>measure algebra</u>

DEF Given $(X_i, \mathcal{B}_i, \mu_i)$, the measure algebras $(\tilde{\mathcal{B}}_i, \tilde{\mu}_i)$ are said to be isomorphic if there is a bijection $\Phi : \tilde{\mathcal{B}}_2 \to \tilde{\mathcal{B}}_1$ that preserves complements, countable unions and countable intersections (as defined above) and $\tilde{\mu}_1(\Phi \tilde{B}) = \tilde{\mu}_2 \tilde{B}$, $\forall \tilde{B} \in \tilde{\mathcal{B}}_2$.

If their measure algebras are isomorphic, the measure spaces are said to be conjugate

Remarks (1) isomorphism $\Rightarrow$ conjugacy

Proof: $\Phi : \tilde{\mathcal{B}}_2 \to \tilde{\mathcal{B}}_1$ via $\Phi(\tilde{B}) = \widetilde{\phi^{-1}(M_2 \cap B)}$

(2) conjugacy $\not\Rightarrow$ isomorphism in general

Trivial example

$\mathbb{X}_1 = \{x\}$, $\mathcal{B}_1 = \{\{x\}, \phi\}$, $\mu_1(\{x\}) = 1$

$\mathbb{X}_2 = \{y, z\}$, $\mathcal{B}_2 = \{\phi, \mathbb{X}_2\}$, $\mu\{\mathbb{X}_2) = 1$

Both measure algebras contain ϕ and another set but a set of measure zero cannot be ommitted from $\mathbb{X}_2$ and have the result mapped bijectively to $\mathbb{X}_1$ [the only zero measure set in $\mathcal{B}_2$ is ϕ.

But: Theorem If $\mathbb{X}_i$ are separable metric and $\mathcal{B}_i$ are Borels we have isomorphism $\Leftrightarrow$ conjugacy.

- Remark on Lebsegue Space: Some books allow a Lebesque space to have a countable set of atoms and it is then isomorphic to $[0,1]$ with Lebesque union a countable set of atoms.

- It is common in Ergodic Theory to restrict to Lebesque spaces.

- A 3rd way to compare probability spaces is via their L^p spaces, most importantly, $L^2(\mu)$

- The first observation is that when X_1 and X_2 are metric spaces with μ_1 and μ_2 Borel measures, then both $L^2(\mu_1)$ and $L^2(\mu_2)$ are separable and thus have a countable basis.

- This means they are isometric unitarily i.e. $\exists$ bijective linear map $W: L^2(\mu_2) \to L^2(\mu_1)$ with

$$\langle W\alpha, W\beta \rangle = \langle \alpha, \beta \rangle \quad \forall \alpha, \beta \in L^2(\mu_2)$$

Thus L^2 alone will not distinguish the probability spaces

- But $L^2(\mu)$ has an additional structure of pointwise multiplication for bounded α and β.

- This turns out to give the tool to classify up to measure conjugacy.

<u>Theorem</u> $(X_i, \mathcal{B}_i, \mu_i)$ are probability spaces.

They are conjugate $\iff \exists$ a bijective linear map $V : L^2(\mu_2) \to L^2(\mu_1)$ such that

(a) $\langle V\alpha, V\beta \rangle = \langle \alpha, \beta \rangle \quad \forall \alpha, \beta \in L^2(\mu_2)$

(b) V and V^{-1} map bounded functions to bounded functions

(c) $(V\alpha)(V\beta) = V(\alpha\beta)$ (pt wise multiplication) whenever α, β are bounded

<u>RK</u>: Recall that $L^2(\mu)$ consists of equivalence classes up to measure zero, so, for example, α bounded means $\exists$ full measure set B and constant K so that $|\alpha(x)| < K$ when $x \in B$.