Isomorphism and conjugacy of mpt

· Corresponding to each notion of equivalence of prob. spaces in the last lecture we have a notion of equivalence for mpt. Here is a schematic for isomorphism of mpt.

$$\begin{tikzcd} M_1 \subseteq \overline{X}_1 \arrow[r, "f_1"] \arrow[r, bend left] \arrow[d, "\phi"'] & \overline{X}_1 \supseteq M_1 \arrow[d, "\phi"] \\ M_2 \subseteq \overline{X}_2 \arrow[r, "f_2"] \arrow[r, bend right] & \overline{X}_2 \supseteq M_2 \end{tikzcd}$$

M_1 and M_2 are full measure $f_i(M_i) \subseteq M_i$

and ϕ is bijective and bi-measure preserving.

<u>DEF</u> $f_i : (X_i, \mathcal{B}_i, \mu_i)$ $i=1,2$ are mpT of probability space

$\exists M_i \in \mathcal{B}_i$, $\mu_i(M_i) = 1$, $f_i(M_i) \subseteq M_i$

and bijective $\phi : M_1 \to M_2$, bi-measure preserving

with $\phi f_1 = f_2 \phi$ on M_1. Then f_1 is isomorphic to f_2 written $f_1 \simeq f_2$

<u>RKS</u>: (1) $\simeq$ is an equivalence relation

(2) $f_1 \simeq f_2 \Rightarrow f_1^n \simeq f_2^n$ $\forall n > 0$

(3) If f_1 and f_2 are invertible, require $f_i(M_i) = M_i$.

Example: $X_1 = [0,1)$ with Borels and Lebesgue measure μ_1

$S: X_1 \to X_1$ is $S(x) = 2x \mod 1$.

$X_2 = \Sigma_2^+$ with Borels, the $(1/2, 1/2)$ - Bernoulli measure μ_2

$\sigma: X_2 \to X_2$ is the shift

$D = \{ \frac{k}{2^n} : 0 \le k \le 2^n \} \subseteq X_1$

$D_2 = \{ \underline{s} \in \Sigma_2 : \underline{s}$ doesn't end with 0^∞ or $1^\infty \} \subseteq X_2$

$\Psi: \Sigma_2^+ \to [0,1]$, $\Psi(\underline{s}) = \frac{s_0}{2} + \frac{s_1}{2^2} + \frac{s_2}{2^3} + \cdots$

$\mu_1(D) = 0 = \mu_2(D_2)$

$\Psi: \Sigma_2^+ - D_2 \to [0,1] - D$ is bijective and measure preserving.

$\sigma_2^{-1}(D_2) = D_2$ so $M_2 = \Sigma_2^+ - D_2$ yields $\mu_2(M_2) = 1$

and $\sigma_2(M_2) = M_2$

$S^{-1}(D) = D$ so $M_1 = \{0,1\} - D$ yields $\mu_1(M_1) = 1$

and $S(M_1) = M_1$

Now on M_2, $\psi\sigma_2 = S\psi$ since

$\psi\sigma_2(\underline{s}) = \psi(s_1 s_2 \ldots) = \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots$

and $S\psi(\underline{s}) = S\left(\frac{s_0}{2} + \frac{s_1}{2^2} + \ldots\right) =$

$2\left(\frac{s_0}{2} + \frac{s_1}{2^2} + \frac{s_2}{2^3} + \ldots\right) \bmod 1$

$= \frac{s_1}{2} + \frac{s_2}{2^2} + \ldots$

so $S \simeq \sigma_2$.

Basic properties are shared by isomorphic maps

Assume $f_1 \simeq f_2$

(1) f_1 is ergodic $\Longleftrightarrow$ f_2 is

(2) f_1 is strong mixing $\Longleftrightarrow$ f_2 is

Proof: (1) will be on next HW

(2) We assume f_2 is mixing and show that f_1 is

Thus, given $A, B \in \mathcal{B}_1$ we must show

$$\mu_1(A \cap f_1^{-n} B) \to \mu(A)\mu(B) \quad (*)$$

Let $A_1 = M_1 \cap A$ and $B_1 = M_1 \cap B$, then $\phi(A_1), \phi(B_1) \in \mathcal{B}_2$

and since f_2 is mixing

$$\mu_2(\phi(A_1) \cap f_2^{-n} \phi(B_1)) \to \mu_2(\phi(A_1))\mu_2(\phi(B_1))$$

$(**)$

Now $\mu_2(\phi(A_1)) = \mu_1(A_1) = \mu_1(A)$ since $A_1 = A \cap M_1$ and $\mu_1(M_1) = 1$ and similary $\mu_2(\phi(B_1)) = \mu_1(B)$.

The issue in working with $\phi(A_1) \cap f_2^{-n}\phi(B_1)$ is that we would like to use $\phi f_1 = f_2 \phi$ to write $f_2^{-n}\phi(B_1) = \phi f_1^{-n} B_1$ the problem is that $f_1(M_1) \subseteq M_1$ and so $f_1^{-n} B_1$ could be not wholly contained in M_1 and so ϕ isn't defined on it. So we need to have a little fact that restricts to the M_i:

<u>FACT:</u> $\forall n \geq 0$ $f_2^{-n} \phi B_1 \cap M_2 = \phi(f_1^{-n}(B_1) \cap M_1)$

<u>Proof:</u> Assume $x \in f_2^{-n}\phi B_1 \cap M_2$ and so $f_2^{n} x \in \phi B_1$ and $\phi^{-1} f_2^{n} x \in B_1$ and $x \in M_2$ and so $f_1^{n}\phi^{-1} x \in B_1$

and so $\phi^{-1}(x) \in f_1^{-n} B_1$ and $x \in M_2 \Rightarrow \phi^{-1}(x) \in M_1$

and so $\phi^{-1}(x) \in f_1^{-n} B_1 \cap M_1$ or $x \in \phi(f_1^{-n} B_1 \cap M_1)$

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For the reverse inclusion we use a trick. Since ϕ is bijective, the FACT is equivalent to

$$f_1^{-n} B_1 \cap M_1 = \phi^{-1}(f_2^{-n} \phi B_1 \cap M_2)$$

let $C = \phi(B_1)$ we get.

$$f_1^{-n} \phi^{-1} C \cap M_1 = \phi^{-1}(f_2^{-n} C \cap M_2)$$

Which is the same form as above with $1 \leftrightarrow 2$, $B_1 \leftrightarrow C$, $\phi \leftrightarrow \phi^{-1}$ and so the argument already given yields the opposite inclusion. Now we have a string of equalities.

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$$
\begin{aligned}
&\mu_2(\phi(A_1) \wedge f_2^{-n}\phi B_1) \\
&= \mu_2(\phi(A_1) \wedge f_2^{-n}\phi B_1 \wedge M_2) && [\mu_2(M_2)=1] \\
&= \mu_2(\phi(A_1) \wedge \phi(f_1^{-n}B_1 \wedge M_1)) && [\text{Little Fact}] \\
&= \mu_2(\phi(A_1 \wedge f_1^{-n}B_1 \wedge M_1)) && [\phi \text{ is bijective}] \\
&= \mu_1(A_1 \wedge f_1^{-n}B_1 \wedge M_1) && [\phi \text{ is m.p.}] \\
&= \mu_1(A_1 \wedge f_1^{-n}B_1) && [\mu_1(M_1)=1]
\end{aligned}
$$

Thus $(**) \Longrightarrow (*)$. $\boxtimes$

Conjugacy of MPT

- Just like conjugacy of probability spaces defined via an isomorphism of the associated measure algebras, we have a notion of conjugate mpt.

- Let $(X_i, \mathcal{B}_i, \mu_i)$ be probability spaces with measure algebras $(\tilde{\mathcal{B}}_i, \tilde{\mu}_i)$. If $\phi: X_1 \to X_2$ is a mpt, it induces a map $\tilde{\phi}^{-1}: (\tilde{\mathcal{B}}_2, \tilde{\mu}_2) \to (\tilde{\mathcal{B}}_1, \tilde{\mu}_1)$ which is well defined on equivalence classes since ϕ is measure pres. (on next HW). It also preserves complements, countable unions and countable intersections. Also, $\tilde{\mu}_1(\tilde{\phi}^{-1}(\tilde{B})) = \tilde{\mu}_2(\tilde{B})$, $\forall \tilde{B} \in \tilde{\mathcal{B}}_2$. Thus it is homomorphism of measure algebras (HW: $\tilde{\phi}^{-1}$ is injective).

DEF: An isomorphism of measure algebras $\Phi : (\tilde{\mathcal{B}}_2, \tilde{\mu}_2) \to (\tilde{\mathcal{B}}_1, \tilde{\mu}_1)$ is a bijection that preserves complements, countable unions and countable intersections and $\tilde{\mu}_1(\Phi(\tilde{B})) = \tilde{\mu}_2(\tilde{B})$, $\forall \tilde{B} \in \tilde{\mathcal{B}}_2$.

DEF $f_i : (X_i, \mathcal{B}_i, \mu_i)$ $i = 1, 2$ are said to be conjugate if there exists a measure-algebra isomorphism $\Phi : (\tilde{\mathcal{B}}_2, \tilde{\mu}_2) \to (\tilde{\mathcal{B}}_1, \tilde{\mu}_1)$

with $\Phi \tilde{f}_2^{-1} = \tilde{f}_1^{-1} \Phi$ with $\tilde{f}_i^{-1}$ defined as on previous page.

Rk: So roughly f_1 and f_2 induce the same map on measure algebras up to a change of coordinates by Φ.

<u>Theorem</u>: If $f_i : (X_i, \mathcal{B}_i, \mu_i)$ 2 are mpt of probability spaces and $f_1 \simeq f_2 \Rightarrow f_1$ is conjugate to f_2

<u>Proof</u>: Let $\phi : M_1 \to M_2$ be the bi-measure preserving bijection in the definition of isomorphic. Define $\Phi : (\overline{\mathcal{B}_2}, \overline{\mu_2}) \to (\overline{\mathcal{B}_1}, \overline{\mu_1})$ by $\Phi(\tilde{B}) = \widetilde{\phi^{-1}(B \cap M_2)}$. Then it is straightforward to check that Φ satisfies the necessary properties in the definition of conjugacy f_1 to f_2. ∎

<u>Remark</u> If each $(X_i, \mathcal{B}_i, \mu_i)$ is a Lebesgue space of each is such that X_i is complete, separable metric and $\mathcal{B}_i$ are the Borels Then the converse is true, namely, conjugacy implies isomorphism.

Spectral Isomorphisms

- Recall a mpt $f : (X, \mathcal{B}, \mu) \circlearrowleft$ induces an operator
  $U_f : L^2(\mu) \to L^2(\mu)$

- **DEF**: $f_i : (X_i, \mathcal{B}_i, \mu_i) \circlearrowleft$ two mpt are spectrally isomorphic if $\exists$ linear operator $W : L^2(\mu_2) \to L^2(\mu_1)$ with

  (1) W invertible

  (2) $\langle W\alpha, W\beta \rangle = \langle \alpha, \beta \rangle$, $\forall \alpha, \beta \in L^2(\mu_2)$

  (3) $U_{f_1} W = W U_{f_2}$

Remarks: (a) (1) and (2) say that W is an isomorphism of the Hilbert spaces

(b) Rewriting (3) as $U_{f_1} = W U_{f_2} W^{-1}$ the operators are conjugate or "similar" and so have the same ~~eige~~ spectrum of eigenvalues.

Theorem: (a) Conjugacy $\Rightarrow$ spectral isomorphic

(b) If f_1 spectral isomorphic to f_2

(i) f_1 is ergodic $\Leftrightarrow$ f_2 is

(ii) f_1 is strong mixing $\Leftrightarrow$ f_2 is

Proof We only do (i). Recall f_1 is ergodic $\Leftrightarrow$ $\alpha \in L^2(\mu)$ $\alpha \circ f = \alpha$ in $L^2(\mu)$ implies α is a constant. In other words, $U\alpha = \alpha$ iff α is a constant. Thus the eigenspace of the eigenvalue 1 for U is the one-dimensional subspace of constants. This property is preserved by spectral isomorphism. ∎

Remark: There are many deep results concerning the spectrum but they require Hilbert space theory which would take a while to develop. We focus next on how to distinguish isomorphism classes via the entropy.