Entropy, con't

- Recall the definition $f \circlearrowleft (X, \mathcal{B}, \mu)$ is a mpt of a probability space

- If $\rho = \{A_1, \ldots, A_k\}$ is a finite partition

$$H(\rho) = -\sum_{i=1}^{k} \mu(A_i) \log \mu(A_i)$$ with the convention $0 \log 0 = 0$

- $$h(f, \rho) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i} \rho \right)$$

- $h(f) = \sup(f, \rho)$ over all finite partitions of X.

<u>Theorem</u>: If $f_i : (X_i, \mathcal{B}_i, \mu_i)$ 2 are conjugate then $h_{\mu_1}(f_1) = h_{\mu_2}(f_2)$ or $h(f_1) = h(f_2)$

<u>Remark</u>: If f_1 is isomorphic to f_2 then they are conjugate and so $h(f_1) = h(f_2)$, Thus entropy is also a isomorphism invariant.

<u>Proof</u> By assumption of conjugacy, $\exists$ an isomorphism of measure algebras $\Phi : (\tilde{\mathcal{B}}_2, \tilde{\mu}_2) \to (\tilde{\mathcal{B}}_1, \tilde{\mu}_1)$ with $\Phi \tilde{f}_2^{-1} = \tilde{f}_1^{-1} \Phi$. Let ρ_2 be a finite partion of X_2, $\rho_2 = \{A_1, \ldots, A_r\}$. Since Φ is onto we may find $\forall i$ $\tilde{B}_i \in \mathcal{B}_1$ with $\Phi(\tilde{A}_i) = \tilde{B}_i$ and pick $B_i \in \tilde{B}_i$ so that $\rho_1 = \{B_1, \ldots, B_r\}$ forms a partition of X_1 (This requires a little work ignoring overlapping sets of measure zero)

Now a typical element of $\bigvee_{i=0}^{n-1} f_1^{-i} \rho_1$ looks like

$\bigcap_{i=0}^{n-1} f_1^{-i} B_{q_i}$ with $q_i \in \{1, \ldots, r\}$ and then

$$\Phi\left(\bigcap_{i=0}^{n-1} \overline{f_2^{-i} A_{q_i}}\right) = \Phi\left(\bigcap_{i=0}^{n-1} \tilde{f}_2^{-i} \tilde{A}_{q_i}\right)$$

$$= \bigcap_{i=0}^{n-1} \tilde{f}_1^{-i} \Phi(\tilde{A}_{q_i}) = \bigcap_{i=0}^{n-1} \tilde{f}_1^{-i} \tilde{B}_{q_i}$$

$$= \bigcap_{i=0}^{n-1} \overline{f_1^{-i} B_{q_i}}$$

~~The~~ Using the conjugacy a basic properties of the action of maps on the measure algebra.

This implies that

$$\mu_2\left(\bigcap_{i=0}^{n-1} f_2^{-i} A_{q_i}\right) = \mu_1\left(\bigcap_{i=0}^{n-1} f_1^{-i} B_{q_i}\right)$$

This happens for every element of $\bigvee_{i=0}^{n-1} f_1^{-i} \rho_1$

and thus for each n $H\left(\bigvee_{i=0}^{n-1} f_1^{-i}\rho_1\right) = H\left(\bigvee_{i=0}^{n-1} f_1^{-i}\rho_2\right)$

and so $h(f_1, \rho_1) = h(f_2, \rho_2)$ and then take

the sup $h(f_1) \geq h(f_2)$ [we started with f_1 above]

But the symmetric argument holds, so $h(f_1) = h(f_2)$ $\blacksquare$

Even though $h(f) < \infty$ is the interesting case, examples with $h(f) = \infty$ happen even when f is a continuous function of a compact metric space.

Example: Let $X = \prod_{i=0}^{\infty} [0,1]$ with the Borel σ-algebra based on the product topology and μ is the product measure using Lebesgue on each $[0,1]$ factor.

Fix $m > 0$, for $1 \le i \le m$ let

$$A_i = \{ \underline{x} \in X : \tfrac{i-1}{m} < x_0 \le \tfrac{i}{m} \}$$
$$= \left(\tfrac{i-1}{m}, \tfrac{i}{m}\right] \times \prod_{i=1}^{\infty} [0,1]$$

Let $P_m = \{A_1, .., A_m\}$ which is a partition of X.

Note that $\mu(A_i) = 1/m$ for all i and so $H(P_m)$

$$= -\sum_{i=1}^{n} \frac{1}{m} \log \frac{1}{m}) = +m \frac{1}{m} \log m = \log(m).$$

Now $\sigma^{-1}(A_i) = [0,1] \times \left(\frac{i-1}{m}, \frac{i}{m}\right] \times \prod_{i=2}^{\infty} [0,1]$

so elements of $\rho_m \vee \sigma^{-1}(\rho_m)$ look like

$$A_{i_0} \cap \sigma^{-1}(A_{i_1}) = \left(\frac{i_0-1}{m}, \frac{i_0}{m}\right] \times \left(\frac{i_1-1}{m}, \frac{i_1}{m}\right] \times \prod_{i=2}^{\infty} [0,1]$$

which has measure $\left(\frac{1}{m}\right)^2$ and so

$$H(\rho_m \vee \sigma^{-1}(\rho_m)) = -\sum_{1 \le i_0, i_1 \le m} \left(\frac{1}{m^2}\right) \log 1/m^2 = 2 \log m$$

elements of $\rho_m \vee \sigma^{-1}\rho_m \vee \cdots \vee \sigma^{-(n-1)}\rho_m$ look like

$$\left(\frac{i_0-1}{m}, \frac{i_0}{m}\right] \times \cdots \times \left(\frac{i_{n-1}-1}{m}, \frac{i_{n-1}}{m}\right] \times \prod_{i=n}^{\infty} [0,1]$$

and have measure $\left(\frac{1}{m}\right)^n$ and so

$$H\left(\bigvee_{i=0}^{n-1} \sigma^{-i}\rho_m\right) = n \log m$$

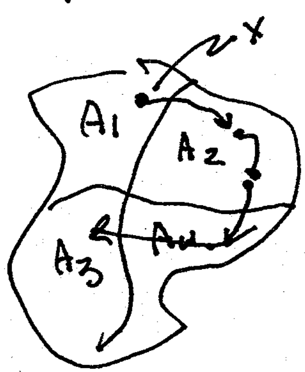
Thus $h(T, \rho_m) = \lim_{n\to\infty} \frac{1}{n} n \log m = \log m.$

and so $h(T) = \sup_\rho h(T,\rho) = \infty.$ ∎

We introduce the main computation tool for entropy, generators, via the itinerary map.

Let $(X, \mathcal{B}(X), \mu)$ be a probabilityspace and $f: X \circlearrowleft$ be mpt which is bijective (so bi-mpt) and

$\rho = \{A_1, \dots, A_k\}$ is a finite partition of X



$\phi: X \to \Sigma_K$ is the itinerary map

$(\phi(x))_j = \ell \iff f^j(x) \in A_\ell$

note $j \in \mathbb{Z}$ so we run the experiment in forward and backward time.

Let $\mathcal{B}(\Sigma_k)$ be the Borels on Σ_k, $\sigma : \Sigma_k \circlearrowleft$ the shift. NOTE that immediately $\phi \circ f = \sigma \circ \phi$. We want to investigate when ϕ yields a conjugacy, or put in another way, the output signals or the results of the experiment completely characterize f acting on X (up to measure zero).

The crucial observation is that

$$x \in \bigcap_{j=w}^{r} f^{-j}(A_{s_j}) \iff f^j(x) \in A_{s_j} \text{ for } j = w, \dots, r \iff \phi(x) \in {}_w[s_w \dots s_r]$$

Thus

$$\phi^{-1}\, {}_w[s_w \dots s_r] = \bigcap_{j=w}^{r} f^{-j}(A_{s_j}) \quad (*)$$

- Since the cylinder sets generate the Borels in Σ_K, ϕ is measurable
- Note that, in general, ϕ is far from injective, then it is constant on partition elements

  eg: $\phi = id$

- Let $\nu = \phi_* \mu$, the push forward of μ to ν defined by $\nu(y) = \mu(\phi^{-1}(y))$
- This is exactly saying that ϕ is measure preserving $\phi : (X, \mathcal{B}(X), \mu) \to (\Sigma_K, \mathcal{B}(\Sigma_K), \nu)$

  but note that we don't know what the image of ϕ looks like in Σ_K

- Now $\nu = \phi_* \mu$ so $\sigma_* \nu = \sigma_* \phi_* \mu$
$= \phi_* \circ f_* \mu = \phi_* \mu = \nu$ since $\phi \circ f = \sigma \circ \phi$
and μ is an invariant measure for f.
- Thus σ is a mpt of $(\Sigma_k, \mathcal{B}(\Sigma_k), \nu)$, but again we emphasize that we know nothing about $\mathrm{spt}(\nu)$

- We now explore the conditions on ρ that allow ϕ to induce a conjugacy between $(X, \mathcal{B}(X), \mu)$ and $(\Sigma_k, \mathcal{B}(\Sigma_k), \nu)$ under f and \sigma

- Recall that on measure algebras ϕ induces

$$\tilde{\phi}^{-1} : (\widetilde{\mathcal{B}(\Sigma_k)}, \tilde{\nu}) \to (\widetilde{\mathcal{B}(X)}, \tilde{\mu})$$

via $\tilde{\phi}^{-1}(\tilde{B}) = \widetilde{\phi^{-1}(B)}$

- As in HW5 #3, $\tilde{\phi}^{-1}$ is injective and from $\phi \circ f = \sigma \circ \phi$ we have $\tilde{\phi}^{-1} \tilde{\sigma}^{-1} = \tilde{f}^{-1} \tilde{\phi}^{-1}$ so to get that $\tilde{\phi}^{-1}$ is a conjugacy, we need that it is onto which requires conditions on ρ.

- Recall from (*) above $\tilde{\phi}^{-1}\,{}_{w}[s_w \ldots s_r] = \bigcap_{j=w}^{r} f^{-j} A_{s_j} \in \bigvee_{i=w}^{r} f^{-i} \rho$ where $[s_w \ldots s_r]$ is arbitrary as is w and r. Thus to have the image of $\tilde{\phi}^{-1}$ be all all $\mathcal{B}(X)$ we must have that all $\bigvee_{i=w}^{r} f^{-i} \rho$ (including $w = -\infty, r = +\infty$) together generate $\mathcal{B}$ after taking the measure classes.

- It is easiest to express this in terms of measure algebras

- DEF: If $\mathcal{C}, \mathcal{D}$ are measure algebras, write $\mathcal{C} \overset{\circ}{\subseteq} \mathcal{D}$ if $\forall C \in \mathcal{C}\ \exists D \in \mathcal{D}$ with $\mu(C \Delta D) = 0$. This happens $\Leftrightarrow$ $\tilde{\mathcal{C}} \subseteq \tilde{\mathcal{D}}$. This write $\mathcal{C} \overset{\circ}{=} \mathcal{D}$ if $\mathcal{C} \overset{\circ}{\subseteq} \mathcal{D}$ and $\mathcal{D} \overset{\circ}{\subseteq} \mathcal{C}$. This happens $\Leftrightarrow$ $\tilde{\mathcal{C}} = \tilde{\mathcal{D}}$

- Thus if $A(\rho)$ is the finite measure algebra for ρ, we have that $\tilde{\phi}^{-1}$ is an isomorphism when

$$\bigvee_{i=-\infty}^{\infty} f^{-i} A(\rho) \overset{\circ}{=} \mathcal{B}(X).$$

- A finite partion ρ or algebra $\mathcal{A}$ is a finite generator for f acting on ~~$\mathcal{B}$~~ $(X, \mathcal{B}, \mu)$

if $$\bigvee_{i=-\infty}^{\infty} f^{-i}\mathcal{A}(\rho) \overset{\circ}{=} \mathcal{B}(X)$$

or $$\bigvee_{i=-\infty}^{\infty} \widetilde{f^{-i}\mathcal{A}} = \widetilde{\mathcal{B}(X)}$$ measure algebras

- We have seen that if $f: (X, \mathcal{B}, \mu) \circlearrowleft$ has a finite generator then f is conjugate to $(\Sigma_k, \mathcal{B}(\Sigma_k), \nu)$ for some k and shift invariant meas ν

- These measures don't have to be Markov or Bernoulli, there are many others.

- A deep result of Krieger says that when $(X, \mathcal{B}, \mu)$ is a Lebesgue space and f is a mpt that is ergodic $\Rightarrow$ it has a finite generator
- Thus in this case, it is conjugate to some $(\Sigma_K, \mathcal{B}(\Sigma_K), \nu)$.