Computation of Entropy

A finite partition $\rho = \{A_1, \dots, A_k\}$ is called a <u>finite generator</u> for the <u>invertible</u> & bi-mpt $f : (X, \mathcal{B}, \mu) \circlearrowleft$ if

$$\bigvee_{n=-\infty}^{\infty} f^{-n} A(\rho) \overset{\circ}{=} \mathcal{B}.$$

<u>Theorem</u> (Kolmogorov-Sinai) If ρ is a finite generator for f as above, $h(f) = h(f, \rho)$ i.e. the entropy of the partition realizes the sup over all partitions and yields the correct entropy

<u>Remark</u>: If f is not invertible, then one requires

$$\bigvee_{n=0}^{\infty} f^{-n} A(\rho) \overset{\circ}{=} \mathcal{B}$$ and then $h(f) = h(f, \rho)$.

Example The two-sided (p, A) Markov shift on Σ_k has entropy $-\sum_{i=1}^{k} \sum_{j=1}^{k} p_i A_{ij} \log A_{ij}$

Computation Let $A_\ell = {}_0[\ell] = \{ \underline{s} \in \Sigma_k : s_0 = \ell \}$

so $\rho = \{ A_0, \dots, A_{k-1} \}$ is a partition of Σ_n. As we have seen previously $\bigcap_{i=w}^{r} \sigma^{-i}(A_{s_i}) = {}_w[s_w \dots s_r]$

and since cylinder sets generate the Borels on Σ_n,

$$\bigvee_{i=-\infty}^{\infty} \sigma^{-i} A(\rho) \overset{\circ}{=} \mathcal{B}$$

So ρ is a finite generator

Thus we need to compute $H(\rho \vee \sigma^{-1}\rho \vee \cdots \vee \sigma^{-(n-1)}\rho)$

for each n.

Let's do a sample case of $n=3$ in Σ_2

so $\rho \vee \sigma^{-1}\rho \vee \sigma^{-2}\rho = \{[000], [001], \ldots, [1,1,1]\}$

Recalling that $\mu({}_0[i_0\, i_1\, i_2]) = p_{i_0} A_{i_0 i_1} A_{i_1 i_2}$

(we index $\vec{p}$ and $\vec{A}$ from 0 to $k-1$. Thus

$$-H(\rho \vee \sigma^{-1}\rho \vee \sigma^{-2}\rho) = \sum_{i_0, i_1, i_2 = 0}^{1} p_{i_0} A_{i_0 i_1} A_{i_1 i_2} \log(p_{i_0} A_{i_0 i_1} A_{i_1 i_2})$$

note

$$= \sum p_{i_0} A_{i_0 i_1} A_{i_1 i_2} (\log p_{i_0} + \log A_{i_0 i_1} + \log A_{i_1 i_2})$$

Terms will come in two types. First terms with a $\log(p_{L_0})$. As an example, let $L_0 = 0$.

There are 4 terms

$$p_0 A_{00} A_{00} \log p_0 + p_0 A_{00} A_{01} \log p_0 + p_0 A_{01} A_{10} \log p_0 + p_0 A_{01} A_{11} \log p_0$$

Now A is a stochastic matrix so $A_{00} + A_{01} = 1 = A_{10} + A_{11}$

so this sum is

$$(p_0 A_{00} + p_0 A_{01}) \log p_0$$

$$= p_0 \log p_0$$

Thus $-H$ has one term $\sum_{L_0=0}^{1} p_{L_0} \log p_{L_0}$

Terms of the second type look like

$$p_{i_0} A_{i_0 i_1} A_{i_1 i_2} \log A_{i_0 i_1} \quad (A)$$

or

$$p_{i_0} A_{i_0 i_1} A_{i_1 i_2} \log A_{i_1 i_2} \quad (B)$$

Again as an example, we examine terms that have a $\log A_{10}$ in it

$$(B) \begin{cases} p_0 A_{01} A_{10} \log A_{10} \\ + p_1 A_{11} A_{10} \log A_{10} \end{cases}$$

$$[p_0 \; p_1] \begin{bmatrix} A_{00} & A_{01} \\ A_{10} & A_{11} \end{bmatrix}$$

$$(A) \begin{cases} + p_1 A_{10} A_{00} \log A_{10} \\ p_1 A_{10} A_{01} \log A_{10} \end{cases}$$

Now, recall $\vec{p} A = \vec{p}$ so $p_0 A_{01} + p_1 A_{11} = p_1$

and $A_{00} + A_{01} = 1$ by stochasticity so this sum is

$$2 p_1 A_{10} \log A_{10}$$

So the second kind of terms added yield

$$2\sum_{i=0}^{1}\sum_{j=0}^{1} p_i A_{ij} \log A_{ij}$$

so
$$H(\rho \vee \tau^{-1}\rho \vee \tau^{-2}\rho) = -\left[\sum_{i=0}^{1} p_i \log p_i + 2\sum_{i=0}^{1}\sum_{j=0}^{1} p_i A_{ij} \log A_{ij}\right]$$

Now for the general case. We use two facts and a factorization as in the above examples.

By $\vec{p}A = \vec{p}$ we have $\sum_{i=0}^{k-1} p_i A_{ij} = p_j$

and $\sum_{i=0}^{k-1} p_{ij} = 1$ both for all j.

$$H(\rho \vee \sigma^{-1}\rho \vee \cdots \vee \sigma^{-(n-1)}\rho) =$$

$$- \sum_{i_0, \ldots, i_{n-1}}^{k-1} p_{i_0} A_{i_0 i_1} \cdots A_{i_{n-2} i_{n-1}} \log(p_{i_0} A_{i_0 i_1} \cdots A_{i_{n-2} i_{n-1}})$$

$$= - \sum_{i=0}^{k-1} p_i \log p_i - (n-1) \sum_{i,j=0}^{k-1} p_i A_{ij} \log A_{ij}$$

Now we showed above that the first term is bounded by $\log k$ and $(n-1)/n \to 1$ as $n \to \infty$

So $H(\sigma) = \lim_{n\to\infty} H\left(\bigvee_{\ell=0}^{n-1} \sigma^{-\ell}\rho\right)/n = - \sum_{i,j=0}^{k-1} p_i A_{ij} \log A_{ij}$

using the Kolmogorov-Sinai Theorem ∎

The next computational tool says roughly that entropy measures exponential growth of possibilities: If the number of possibilities for outcomes of an experiment grow in time at a polynomial rate, then the entropy is zero

<u>Some notation:</u> For a finite set S, $\#(S)$ is the cardinality of S

A sequence a_n is "big oh n^k" written $a_n = O(n^k)$ for some $k > 0$ if $\exists$ constant C so that $a_n \leq C n^k$, $\forall n$.

Recall For a finite partition ρ, ρ^n is short hand for $\rho^n = \bigvee_{i=0}^{n} f^{-i} \rho$ with f being implicit.

<u>Lemma</u> Let $f:(X, \mathcal{B}, \mu)$ be an invertible, bi-mpt

(1) If for all partitions ρ, $\#(\rho^n) = O(n^k)$ for some k,

then $h_\mu(f) = 0$

(2) If f has a generator ρ with $\#(\rho^n) = O(n^k)$

for some $k \Rightarrow h_\mu(f) = 0$

<u>Proof</u> We showed previously that if $\#(\rho) = \ell$ then $H(\rho) \leq \log(\ell)$. Thus if $\#(\rho^n) \leq C n^k$

$$h(f, \rho) = \lim_{n\to\infty} \frac{1}{n} H(\rho^n) \leq \lim_{n\to\infty} \frac{\log(Cn^k)}{n}$$

$$= \lim_{n\to\infty} \frac{\log C + k \log n}{n} = 0$$

and (1) and (2) follow from definitions.

DEF: $f : (X, \mathcal{B}, \mu)$ is called <u>finite order</u> if for some p, $f^{-p} = id$ $(p > 0)$

CORR If f is finite order, $h_\mu(f) = 0$

Proof Since $f^{-p}(\rho) = \rho$ for any partition ρ

$\rho^{p+l} = \rho^{p}$ for any $l \geq 0$ since

$$\rho^{p+l} = \bigvee_{i=0}^{p-1} f^{-i}(\rho) \vee \bigvee_{i=p}^{l-1} f^{-i}(\rho)$$

$$= \bigvee_{i=0}^{p-1} f^{-i}(\rho) \vee \bigvee_{i=0}^{l-1-p} f^{-i}(\rho)$$

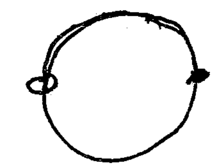
$$= \cdots = \bigvee_{i=0}^{p-1} f^{-i}(\rho)$$

Thus $\# \rho^{p+l} = \# \rho^{p}$ so $\# \rho^{n}$ is $O(1)$ so $h(f) = 0$.

<u>Example</u>: $R_{p/q} : S^1 \circlearrowleft$ preserves Lebesque and

$(R_{p/q})^{q} = id$ so $h(R_{p/q}) = 0$

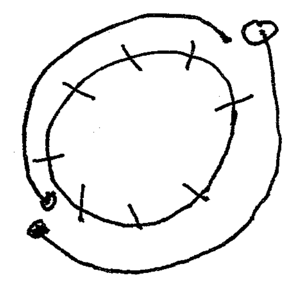
<u>Example</u> $R_\omega : S^1 \circlearrowleft$ $\omega \notin \mathbb{Q}$. Let



$\rho = \{ [0, 1/2), [1/2, 1) \}$

Since $o(0, R_\omega)$ and $o(1/2, R_\omega)$ are both dense any half open interval in the circle will be a countable union of sets in $\bigvee_{i=-\infty}^{\infty} R_\omega^{-i} \rho$

and so ρ is a generator.



Since $R_\omega^{-n}(0)$ and $R_\omega^{-n}(1/2)$ will fall into the interiors of elements of ρ^{n-1}

we have that $\#(\rho^n) = \#(\rho^{n-1}) + 2$

and $\#(\rho) = 2$ so $\#(\rho^n) = 2n$ and so

$h(R_w) = 0.$

Example In (Σ_2, σ) with $\rho = \{{}_0[0], {}_0[1]\}$

$\#(\rho^n) = 2^n$ grows exponentially on

we know (Σ_2, σ) has measures of positive

entropy.