Extension Theorems

- To get further in presenting examples we need some theory that is analogous with going from a base to a topology or from a basis to a vector space

- **DEF:** $\mathcal{S}$ a collection of subsets of X is a <u>semi-algebra</u> if

  (1) $\emptyset \in \mathcal{S}$

  (2) $A, B \in \mathcal{S} \Rightarrow A \cap B \in \mathcal{S}$

  (3) $A \in \mathcal{S} \Rightarrow X - A = \bigcup_{i=1}^{n} E_i$ for $E_i \in \mathcal{S}$ pairwise disjoint

Example All subintervals of [0,1] of the form $[0, b]$ and $[a, b]$ with $0 \le a < b \le 1$ form a semi-Algebra

DEF A collection $\mathcal{A}$ of subsets of X is an <u>algebra</u> if

(1) $\emptyset \in \mathcal{A}$
(2) $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$
(3) $A \in \mathcal{A} \Rightarrow X - A \in \mathcal{A}$.

Theorem 1 Each semi algebra generates a unique algebra

Proof $\mathcal{A} =$ all subsets that can be written as $E = \bigcup_{i=1}^{n} A_i$ with $A_i \in \mathcal{S}$, pairwise disjoint

DEF: Let $\mathcal{S}$ be a semi algebra $I : \mathcal{S} \to \mathbb{R}^+$ is finitely additive if for $E_1, \ldots, E_n \in \mathcal{S}$ pairwise disjoint $\Rightarrow I(\cup E_i) = \sum I(E_i)$ and countably additive if $E_1 \ldots \in \mathcal{S}$ pairwise disjoint and

$$I(\cup E_i) = \sum I(E_i)$$

Therem 2 additive and countably additive functions on a semi algebra extend uniquely to the algebra with the same additivity properties

DEF Given and Algebra $\mathcal{A}$ since all subsets of X is a σ-algebra it makes sence to define the sigma-algebra $\mathcal{B}(\mathcal{A})$ as the smallest σ-algebra the contains $\mathcal{A}$.

Theorem 3: Let $\mathcal{A}$ be an algebra and $I_0 : \mathcal{A} \to \mathbb{R}^1$ is countable additive and $I(\mathbb{X}) = 1 \Rightarrow$ there is a unique probability measure on $(\mathbb{X}, \mathcal{B}(\mathcal{A}))$ that extends I.

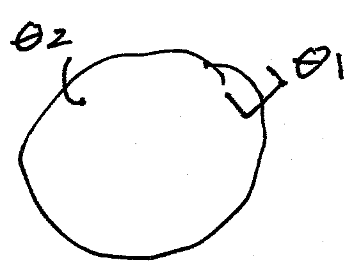
Theorem 4 If $\mathcal{S}$ is a semi-algebra and $I : \mathcal{S} \to \mathbb{R}^1$ is countabley additive and $\sum_{i=1}^{n} I(E_i) = 1$ when $\mathbb{X} = \coprod_{i=1}^{n} E_i$ disjoint union and $E_i \in \mathcal{S} \Rightarrow$ it can be uniquely extended to a probability measure on $\mathcal{B}(\mathcal{A}(\mathcal{S}))$

Examples

(1) Subintervals of $[0,1]$ of the form $[0,b]$ and $(a,b]$ is a semi-algebra and define $I([0,b]) = b$, $I((a,b]) = b-a$

$\Rightarrow$ it can be uniquely extended yielding Lebesque measure on the Borel σ-algebra

(2) $X = S^1$, Subintervals of the form $[\theta_1, \theta_2)$ where θ_2 is counter clockwise from θ_1



This is an algebra and define $I([\theta_1, \theta_2)) = \theta_2 - \theta_1$ This extends to what we call Lebesque measure on the Borels.

Note: This is also the induced measure from $S^1 = \mathbb{R}/\mathbb{Z}$

While we are at it, another useful theorem allows you to go from finitely additive on an algebra to countably additive

<u>Theorem 5</u> Let $I: \mathcal{A} \to \mathbb{R}^+$ be finitely additive on the algebra $\mathcal{A}$; if for every decreasing sequence of elements of $\mathcal{A}$, $E_1 \supset E_2 \supset \dots$ such that $\bigcap_{n=1}^{\infty} E_n = \phi$ we have $I(E_n) \to 0 \Rightarrow I$ is countably additive

<u>References</u>

<u>Books</u>

Parthasarathy, Introduction to Probability and measure, 1977

Kingman and Taylor, Introduction to measure and probability, 1966

the first application is

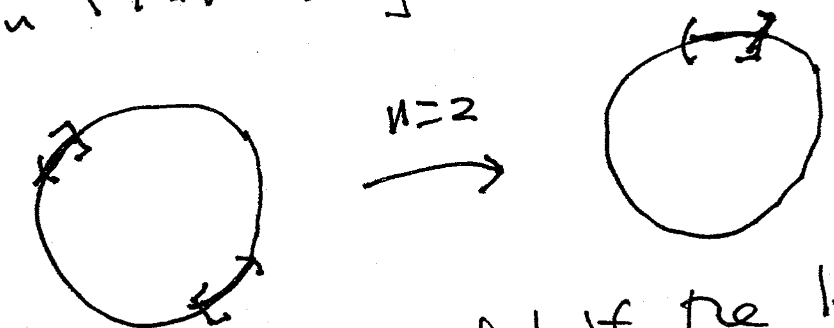
Theorem 6: Say $f : (X_1, \mathcal{B}_1, \mu_1) \to (X_2, \mathcal{B}_2, \mu_2)$ is a transformation of probability spaces. Let $\mathcal{S}_2$ be a semi-algebra that generates $\mathcal{B}_2$ and for each $A_2 \in \mathcal{S}_2$, $f^{-1}(A_2) \in \mathcal{B}_1$ and $\mu_1(f^{-1}(A_2)) = \mu_2(A_2) \Rightarrow f$ is measure preserving

Application S^1 with Lebesque is generated by the algebra $\{[\theta_1, \theta_2)\}$. Let $R_w : S^1 \to S^1$ be $\theta \to \theta + w$. Then $\theta_2 - \theta_1 = \mu([\theta_1, \theta_2))$ and

$$\mu(R_w[\theta_1, \theta_2)) = \mu([\theta_1 + w, \theta_2 + w)) = \theta_2 + w - (\theta_1 + w)$$

$$= \theta_2 - \theta_1 = \mu([\theta_1, \theta_2)).$$ So R_w preserves Lebesque measure

Application: S^1 with Lebeseque measure, $f: S^1 \to S^1$ via $\theta \to n\theta \mod 1$ [or if treating $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ then $f(z) = z^n$] then f preserves Lebesque measure



Two preimages of half the length.

Application: Let $X = [0,1]^n \subseteq \mathbb{R}^n$ with Lebesque measure. $f: X \to X$ is a orientation preserving C^1-diffeomorphism. f is area preserving $\iff \det(Df) \equiv 1$

Proof Since f is bicontinuous it is bi-measurable on the Borel σ-algebra.

Also note that if $\mu(f(A)) = \mu(A)\ \forall A \in \mathcal{B}$

$\Rightarrow \mu(f^{-1}(A)) = \mu(f(f^{-1}(A))) = \mu(A)$, so f is measure preserving.

Let $\mathcal{S}$ be the collection of all rectangles in $\overline{X}$ where we allow sides to be missing. Then $\mathcal{S}$ is a semi-algebra. Now for any $A \in \mathcal{B}$, $\mu(A) = \int_A 1\, d\mu$ using the Lebesque integral. Now using the Reiman integral change of variables formula with $R \in \mathcal{S}$ a rectangle

$$\mu(f(R)) = \int_{f(R)} 1\, d\mu = \int_R |Df(x)|\, d\mu(x) \qquad (*)$$

So if $|Df(x)| \equiv 1$, $\mu(f(R)) = \mu(R)$ using equality of Reimann and Lebesque intervals on Rectangles and there diffeomorphic images. Conversely, if $|Df(x)| > 1$ since f is C^1, $|Df(x)|$ is continuous, so on some rectangle R $|Df(x)| > 1$ for $x \in R$ and then by $(*)$ $\mu(f(R)) \neq \mu(R)$ $\boxed{}$

A similar result is true for $\mathbb{R}^n$ but now we have Lebesgue measure with infinite total mass

Example (Henon map)

$$f(x,y) = (1 - ax^2 + y,\ bx)$$

$$Df = \begin{bmatrix} -2ax & 1 \\ b & 0 \end{bmatrix} \quad \text{so } \det(Df) \equiv -b$$

So $f(x,y) = (1 - ax^2 + y,\ -x)$ is area preserving and in fact is a diffeomorphism.