Topological Entropy

- There is a notion of entropy which just depends on topology but is closely connected to the measure theoretic entropy.

- If $f : X \to X$ is a homeomorphism of a compact metric space $h_{top}(f)$ is its topological entropy (to be defined)

- VARIATIONAL PRINCIPLE:

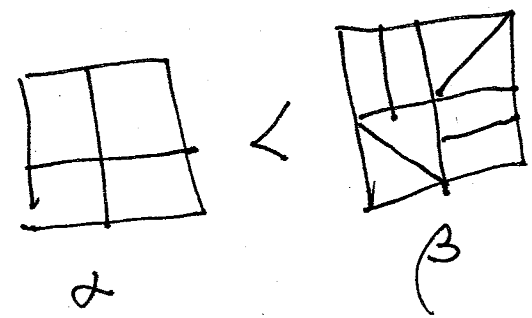
$$h_{top}(f) = \sup\{h_\mu(f) : \mu \text{ is an invariant Borel probability measure for } f\}$$

- Example: $\sigma : \Sigma_M^2$ for a subshift of finite type, $h_{top}(\sigma) = \log \lambda = h_\mu(\sigma)$ for Parry measure.

First some definitions analagous to partitions

- $\alpha = \{U_1, U_2, \dots\}$ is an open cover of X if $X = \bigcup U_i$.
- If X is compact, recall that there is always a finite subcover $\{U_{n_1}, \dots, U_{n_k}\}$ with $X = \bigcup_{i=1}^{k} U_{n_i}$
- The join of α with $\beta = \{V_1, \dots\}$ is $\{U_i \cap V_j : U_i \in \alpha, V_j \in \beta\} = \alpha \vee \beta$
- notice that $\alpha \vee \beta$ is also an open cover of X.
- $\bigvee_{i=1}^{n} \alpha_i$ is defined similarly

- β is a refinement of α, written $\alpha < \beta$ if every member of β is a subset of a member of α



- so $\alpha < \alpha \vee \beta$

- $f : X \to X$ is continuous, onto

$\Rightarrow f^{-1}\alpha := \{ f^{-1}(U_i) : U_i \in \alpha \}$

is also an open cover since inverse images of open sets are open

- FACTS: (1) $f^{-1}(\alpha \vee \beta) = f^{-1}(\alpha) \vee f^{-1}(\beta)$

  (2) $\alpha < \beta \Rightarrow f^{-1}\alpha < f^{-1}\beta$

- **DEF:** α is an open cover then $N(\alpha)$ is the number of elements in a subcover having **least** cardinality.

- **DEF:** The entropy of α is $H(\alpha) = \log N(\alpha)$

  (**not** $N(\alpha) \log N(\alpha)$)

- Properties

  (1) $H(\alpha) \geq 0$

  (2) $H(\alpha) = 0 \Leftrightarrow N(\alpha) = 1 \Leftrightarrow X \in \alpha$

(3) $\alpha < \beta \Rightarrow H(\alpha) \le H(\beta)$

(4) $H(\alpha \vee \beta) \le H(\alpha) + H(\beta)$

<u>Proof:</u> Let $\{U_1, \ldots, U_{N(\alpha)}\}$ be a subcover of α of minimum cardinality and $\{V_1, \ldots, V_{N(\beta)}\}$ for β.
Then $\{U_i \wedge V_j : 1 \le i \le N(\alpha),\ 1 \le j \le N(\beta)\} = \gamma$ has cardinality $N(\alpha)N(\beta)$ and $\alpha \vee \beta < \gamma$
and so $N(\alpha \vee \beta) \le N(\gamma) = N(\alpha)N(\beta)$, now take logs. ☐

(5) If $f : X \to X$ is onto then $H(f^{-1}\alpha) = H(\alpha)$

<u>Proof:</u> Let $\{U_1, \ldots, U_{N(\alpha)}\}$ be a sub cover of min card $\Rightarrow$ $\{f^{-1}(U_1), \ldots, f^{-1}(U_{N(\alpha)})\}$ is a subcover of $f^{-1}(\alpha)$ so $N(f^{-1}\alpha) \le N(\alpha)$.
Since f is onto, if

$\{f^{-1}A_1, \ldots, f^{-1}A_{N(f^{-1}\alpha)}\}$ is a subcover of $f^{-1}\alpha$ of minimum cardinality then $\{A_1, \ldots, A_{N(f^{-1}\alpha)}\}$ covers X, so $N(\alpha) \leq N(f^{-1}\alpha)$ ∎

DEF Given continuous, onto $f: X \to X$ of the compact metric space X and α is an open cover the. topological entropy of α under f is

$$h(f, \alpha) = \lim_{n \to \infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha \right)$$

where we have to show the limit exists.

<u>Theorem</u>: With the hypothesis above

$a_n = H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right)$ is subadditive

$(a_{n+k} \leq a_n + a_k)$ and so $h(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} a_n$ exists.

<u>Proof</u>

$$
\begin{aligned}
a_{n+k} &= H\left(\bigvee_{i=0}^{n+k-1} f^{-i}\alpha\right) \\
&= H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha \vee \bigvee_{i=n}^{n+k-1} f^{-i}\alpha\right) \\
&\leq H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) + H\left(\bigvee_{i=n}^{n+k-1} f^{-i}\alpha\right) \quad (\text{by } (4)) \\
&= H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) + H\left(f^{-n}\bigvee_{j=0}^{k-1} f^{-j}\alpha\right) \\
&= H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) + H\left(\bigvee_{j=0}^{k-1} f^{-j}\alpha\right) \quad (\text{by } (5)) \\
&= a_n + a_k
\end{aligned}
$$

<u>Example</u>: We have essentially seen this before: $X = \Sigma_2$, $f = \sigma$, $\alpha = \{{}_0[0], {}_0[1]\}$

Then $f^{-i}(\alpha) = \sigma^{-i}(\alpha) = \{{}_i[0], {}_i[1]\}$

$$\bigvee_{i=0}^{n-1} \sigma^{-i}(\alpha) = \{[B] : B \text{ is a block of length } n\}$$

So $H\left(\bigvee_{i=0}^{n-1} \sigma^{-i}\alpha\right) = \log(2^n) = n \log 2$

So $h(\sigma, \alpha) = \lim_{n\to\infty} \frac{1}{n} n \log 2 = \log 2$

In general, in Σ_k, $\alpha = \{{}_0[0], \ldots, {}_0[k-1]\}$

$\Rightarrow h(\sigma, \alpha) = \log k.$

Facts continued

(6) $h(f,\alpha) \geq 0$

(7) $\alpha < \beta \Rightarrow h(f,\alpha) \leq h(f,\beta)$

<u>Proof</u> $\alpha < \beta \Rightarrow \bigvee_{i=0}^{n-1} f^{-i}\alpha \leq \bigvee_{i=0}^{n-1} f^{-i}\beta$ so by (3)

$H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) \leq H\left(\bigvee_{i=0}^{n-1} f^{-i}\beta\right)$, divide by n, let $n \to \infty$. ∎

(8) $h(f,\alpha) \leq H(\alpha)$

<u>Proof</u>: By (4) $H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) \leq \sum_{i=0}^{n-1} H(f^{-i}\alpha) = \sum_{i=0}^{n-1} H(\alpha)$

$= nH(\alpha)$ using (5), divide by n, let $n \to \infty$. ∎

<u>DEF</u>: $h(f) = \sup_{\alpha} h(f,\alpha)$ over all open covers of X is the definition of topological entropy.

(9) $h(f) \geq 0$ but $h(f) = \infty$ is possible

(10) By (7), we may also take the sup over finite covers

(11) $h(id) = 0$

(12) Y is compact, $f(Y) = Y \Rightarrow h(f|_Y) \leq h(f)$

<u>DEF</u>: X_i are compact, $f_i : X_i \to X_i$ onto

$\phi : X_1 \to X_2$ onto, continuous. If

$$\begin{array}{ccc} X_1 & \xrightarrow{f_1} & X_1 \\ \phi \downarrow & & \downarrow \phi \\ X_2 & \xrightarrow{f_2} & X_2 \end{array}$$

commutes, (X_1, f_1) is topologically semiconjugate to (X_1, f_2)

If ϕ is a homeomorphism, then (X_1, f_1) is topologically conjugate to (X_2, f_2)

Theorem If (X_1, f_1) is topologically semiconjugate to (X_2, f_2) $\Rightarrow h(f_1) \geq h(f_2)$. If (X_1, f_1) topologically conjugate to $(X_2, f_2) \Rightarrow h(f_1) = h(f_2)$

Proof: Let α be an open cover of X_2

$$h(f_2, \alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f_2^{-i}\alpha\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\phi^{-1} \bigvee_{i=0}^{n-1} f_2^{-i}\alpha\right) \quad \text{by 15)}$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} \phi^{-1} f_2^{-i}\alpha\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f_1^{-i}\phi^{-1}\alpha\right)$$ by the semiconjugacy

$$= h(f_1, \phi^{-1}\alpha)$$

Thus every partition of X_2 yields one for X_1

so : $h(f_1) \geq h(f_2)$. If ϕ is a homeomorphism, Run the argument with $\phi = (\phi^{-1})^{-1}$ yields $h(f_1) \leq h(f_2)$

so $h(f_1) = h(f_2)$

<u>Theorem</u> If $f : X \circlearrowleft$ is a homeomorphism of the compact metric space X then $h(f) = h(f^{-1})$

Proof:

$$h(f,\alpha) = \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(f^{n-1}\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right)\right) \quad \text{by (5)}$$

$$= \lim_{n\to\infty} \frac{1}{n} H\left(\bigvee_{i=0}^{n-1} f^{i}\alpha\right)$$

$$= h(f^{-1},\alpha)$$

Take sups ∎

- NOTE That this all never used the metric just the compactness
- We need a computational tool like the generator theorem which is easier with a second, equivalent definition of the topological entropy.