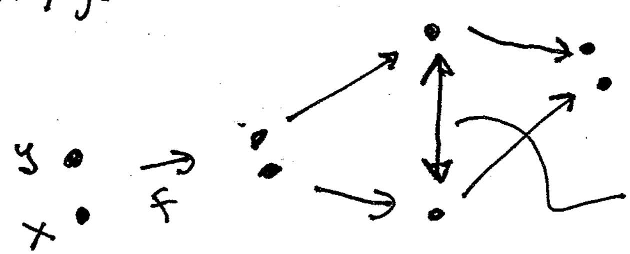
Bowen's definition of Topological Entropy

- Let X be compact with metric d (the theory works without compactness in many places) and $f: X \to X$ is continuous and onto

- For $x, y \in X$, $n \in \mathbb{N}$ Let

$$d_n(x,y) = \max\{ d(f^k(x), f^k(y)) : 0 \le k \le n-1 \}$$



$d_3(x,y) = d_4(x,y)$

- So $d_n(x,y)$ computes the maximum distance between the orbit of x and the orbit of y over the first $(n-1)$ iterates.

<u>NOTATION</u>: For a metric ρ let $B_\varepsilon(x,\rho) = \{y : \rho(x,y) < \varepsilon\}$.

<u>FACTS</u> (1) d_n is a metric for each n

(2) $d_1 = d$

(3) $d_n(x,y) \geq d_{n-1}(x,y)$ and so $B_\varepsilon(x, d_n) \subseteq B_\varepsilon(x, d_{n-1})$

(4) All the d_n yield the same open sets in X

(5) If $y \in B_\varepsilon(x, d_n)$ then $d(f^k(x), f^k(y)) < \varepsilon$ for $k = 0, \ldots, n-1$. If $y \notin B_\varepsilon(x, d_n) \Rightarrow \exists\, k \leq n-1$ with $d(f^k(x), f^k(y)) \geq \varepsilon$.

Fix n an $\varepsilon > 0$

DEF: $S \subseteq X$ is called <u>(n,ε)-spanning</u> if
$\forall x \in X$, $\exists y \in S$ with $d_n(x,y) < \varepsilon$

In other words $X = \bigcup_{y \in S} B_\varepsilon(y, d_n)$ so for all $x \in X$

there is a $y \in S$ with $d(f^k(x), f^k(y)) < \varepsilon$ $k = 0, \dots, n-1$

<u>Remark</u> By compactness we can assume S is finite

DEF: $\mathcal{S}(n,\varepsilon) = \min\{\#S : S \text{ is } (n,\varepsilon)\text{-spanning}\}$

(Recall $\#S$ = cardinality of S).

A kind of dual notion is next

DEF: $K \subseteq X$ is <u>(n,ε)-separated</u> if $x \neq y$ and $\cancel{(x,y)}$ $x, y \in K \Rightarrow d_n(x,y) \geq \varepsilon$.

In other words, for $x \neq y$ in K, $\exists\, k$ with $d(f^k(x), f^k(y)) \geq \varepsilon$.

Remark once again by compactness we can find finite K

DEF $K(n,\varepsilon) = \max\{\#K : K \text{ is } (n,\varepsilon)\text{-separated}\}$

DEF: Let $\mathrm{diam}_n(U)$ be the diameter of U with respect to d_n, so $\mathrm{diam}_n(U) = \sup\{d_n(x,y) : x, y \in U\}$.

Let $\mathcal{U}$ be an open cover of X by sets with d_n-~~D~~diameter $< \varepsilon$ and

$$C(n,\varepsilon) = \min\{\#\mathcal{U} : \mathcal{U} \quad \}$$

Remark Since $d_n(x,y) < \varepsilon$ iff x and y stay close to each other for $n-1$ iterates, each of the above quantities is a way to count the number of different orbits of length n.

Supiisingly, all these measures are NOT related.

Lemma $\mathcal{C}(n, 2\varepsilon) \le \mathcal{S}(n,\varepsilon) \le \mathcal{K}(n,\varepsilon) \le \mathcal{C}(n,\varepsilon)$

Proof: Fix n and $\varepsilon > 0$. Find a finite (n,ε)-spanning set S and use its points as centers for d_n balls of radius 2ε. This gives an open cover of X. Now there might be a smaller (n,ε) cover but in any event we have $\mathcal{C}(n, 2\varepsilon) \le \mathcal{S}(n,\varepsilon)$. Now S is also (n,ε)-separated but allowing for the possibility that a few more (n,ε) points might fit in X,

$\mathcal{S}(n,\varepsilon) \le \mathcal{K}(n,\varepsilon)$.

Finally, let $\mathcal{U}$ be an open cover with sets having d_n diameter $< \varepsilon$ and let $\mathcal{U}$ realize the min so

$$\#\mathcal{U} = \mathcal{C}(n,\varepsilon)$$

we claim there cannot be an (n,ε)-separated set B with $\#B > \#\mathcal{U}$ since by the pigeonhole principle that would mean two distinct points $b, b' \in B$ are in the same $U \in \mathcal{U}$ and so $d_n(b,b') < \varepsilon$, a contradict.

Thus $\mathcal{K}(n,\varepsilon) \leq \mathcal{C}(n,\varepsilon)$. $\blacksquare$

<u>DEF</u>:

$$h(f,\varepsilon) = \lim_{n\to\infty} \frac{\log(C(n,\varepsilon))}{n}$$

where we have to show the limit exists

<u>Lemma</u> The limit exists

<u>Proof</u> We show that $a_n = \log(C(n,\varepsilon))$ is subadditive.

Proof: If $U \subseteq X$ has $\mathrm{diam}_m(U) < \varepsilon$ and $V \subseteq X$ has $\mathrm{diam}_k(V) < \varepsilon$ then $U \wedge f^{-m}V$ has d_{m+k} diam $< \varepsilon$. Thus if $\mathcal{U}$ realizes $\mathcal{C}(m, \varepsilon)$ and $\mathcal{V}$ realizes $\mathcal{C}(k, \varepsilon)$ the cardinality of the set of all $U \wedge f^{-m} V$ is $\leq \mathcal{C}(m,\varepsilon) \cdot \mathcal{C}(k,\varepsilon)$ and the set of all $U \wedge f^{-m} V$ provids a $(m+k, \varepsilon)$-cover so

$$\mathcal{C}(m+k, \varepsilon) \leq \mathcal{C}(m,\varepsilon) \cdot \mathcal{C}(k,\varepsilon)$$

or taking logs $a_{m+k} \leq a_m + a_k$ ∎

Remark: As ε decreases $\mathcal{C}(n, \varepsilon)$ does not decrease so we have

DEF $$h(f) = \lim_{\varepsilon \to 0} h(f, \varepsilon) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log \mathcal{C}(n,\varepsilon)}{n}$$

is Bowen's topological entropy.

Remark: By the lemma on page 6 we also have

$$h(f) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log(\mathcal{K}(n,\varepsilon))}{n} = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{\log(\mathcal{S}(n,\varepsilon))}{n}$$

In various situations one of the definitions may be better for computations.

Example: $f:(X,d)\to(X,d)$ is called an isometry if $\forall x,y\in X$, $d(f(x),f(y)) = d(x,y)$ i.e. it preserves distances.

Prop: If f is an isometry, then $h(f)=0$

Proof It follows immediately that $d_n = d_1 = d$ for all n. Thus if K is finite and ε-separated for $n=0$ then it is (n,ε)-separated for all n.

Thus $K(n,\varepsilon) = K(1,\varepsilon)$ for all n and so

$$h(f) = \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{\log(K(1,\varepsilon))}{n} = \lim_{\varepsilon \to 0} 0 = 0 \quad \blacksquare$$

__Example__: (a) $f : S^1 \circlearrowleft$ via $\theta \mapsto \theta + \alpha$ is an isometry so $h(f) = 0$ with respect to usual metric

(b) $f : \mathbb{T}^2 \circlearrowleft$ via $(\theta_1, \theta_2) \to (\theta_1 + \alpha_1, \theta_2 + \alpha_2)$ mod 1 in both components, is an isometry with respect to the usual metric so $h(f) = 0$.

__Example__ Let G be a compact, metric group for $g_0 \in G$

Then $f_{g_0}(g) = g + g_0$ has $h(f_{g_0}) = 0$ with respect to the metric defined by $\delta(x,y) = \int_G d(gx, gy)\, d\gamma(g)$

where γ is Haar measure.

Pf Since $(f_{g_0})_* \gamma = \gamma$ i.e. f_{g_0} preserves Haar measure

(Recall this is the definition of Haar measure

$$\delta(f_{g_0}x, f_{g_0}y) = \int_G d(g(x+g_0), g(y+g_0))\, d\gamma$$

$$= \int_G d(gx, gy)\, d\gamma$$

$$= \delta(x,y)$$ using the change of variables theorem.

Thus f_{g_0} is an isometry w.r.t. δ so $h(f)=0$

NOTE: This seems to depend on the metric but the next result is that the entropy defined by the metric is independent of the choice of metric on a compact X.