Bowen's Topological Entropy, cont.

- At this point Bowen's definition of topological entropy uses the metric, but we would like to eliminate that dependence.
- We need to review some topology of metric spaces.
- Fix a metric d on X.
- U is an open set if $\forall x \in U\ \exists \varepsilon > 0$ with $B_\varepsilon(x) \subseteq U$

- The topology induced by d is the collection of its open sets.
- Two metrics d and d' are <u>equivalent</u> if they define the same topology OR U is d-open $\Leftrightarrow$ U is d' open.

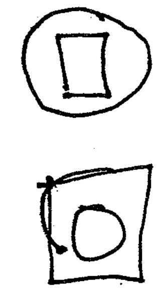
FACT: TFAE

(1) d is equivalent to d'

(2) $\forall x \; \forall \varepsilon > 0$

(a) $\exists \delta$ with $B_\delta(x, d) \subseteq B_\varepsilon(x, d')$

and (b) $\exists \delta'$ with $B_{\delta'}(x, d') \subseteq B_\varepsilon(x, d)$

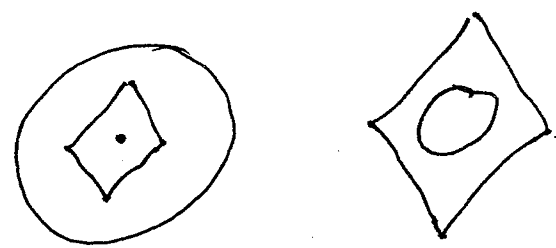


Example on $\mathbb{R}^2$ let

$$d(x, x') = \sqrt{(x_1 - x_1')^2 + (x_2 - x_2')^2}$$

$$d'(x, x') = |x_1 - x_1'| + |x_2 - x_2'|$$

are equivalent metrics



- For two metric spaces $(\mathbb{X}, d)$, $(\mathcal{Y}, \delta)$

(a) $f : \mathbb{X} \to \mathcal{Y}$ is continuous at x

if $\forall \varepsilon > 0\ \exists \delta > 0$ so that

$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$$

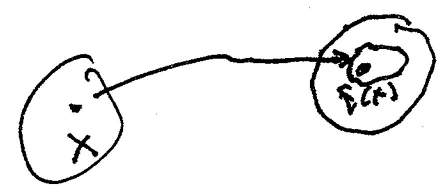
(b) f is uniformly continuous if

$\forall \varepsilon > 0\ \exists \delta > 0\ \forall x$

$$d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon.$$

<u>FACT</u> f is cont at $x \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0$ so that

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$



f is unif cont $\Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0$ so that $\forall x$

$$f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$$

The next fact follows from the definitions just by translations

<u>FACT</u> d is equivalent to d' $\Longleftrightarrow$ $id: (X,d) \to (X,d')$

is a homeomorphism

We need a theorem from topology

<u>Theorem</u> If $f: (X,d) \to (Y,\delta)$ is continuous and (X,d) is compact $\Rightarrow$ f is uniformly continuous

Putting the pieces together.

<u>FACT</u>: If (X,d) is compact and d is equivalent to d'

then $id: (X,d) \to (X,d')$ is bi-uniform homeomorphism

and so $\forall \varepsilon > 0 \ \exists \delta > 0 \ \exists \delta' > 0$ so that $\forall x \in X$

$B_\delta(x,d) \subseteq B_\varepsilon(x,d')$ and $B_{\delta'}(x,d') \subseteq B_\varepsilon(x,d)$

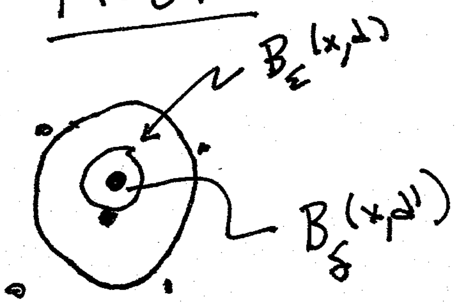
<u>Remark</u>: (1) If d and d' are uniformly equivalent as in the previous fact, then for each n, d_n and d_n' are also with the same δ and δ'

(2) We write $\delta(\varepsilon)$ and $\delta'(\varepsilon)$ to denote the dependence

Note that as $\varepsilon \to 0$, $\delta(\varepsilon) \to 0$ and $\delta'(\varepsilon) \to 0$

<u>Theorem</u>: If X is compact with equivalent metrics d and d' then $h(f)$ computing using Bowen's definition gives the same result using d and d'.

<u>Proof</u>: Let K be d_n-(n,ε) separated



then it is also d_n'-$(n,\delta(\varepsilon))$ separated

Taking the max over all such K yields

which yields

$$\mathcal{K}(n,\varepsilon) \le \mathcal{K}'(n,\delta(\varepsilon))$$

$h(f,\varepsilon) \le h'(f,\delta(\varepsilon))$ and letting $\varepsilon \to 0$ and so $\delta(\varepsilon) \to 0$ we get $h(f) \le h'(f)$. Reversing the roles of d and d' yields $h'(f) \le h(f)$ $\blacksquare$

- We next show that Bowen's top. ent. is a conjugacy invariant. We showed this for the open set definition already so will follow from the eventual result that the open set and Bowen def yield the same result, but the direct proof is instructive

Theorem Assume $f: X \to X$, $g: Y \to Y$ continuous onto maps of compact metric spaces (X,d) and (Y,d') and $\exists$ homeomorphism ϕ with
$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \phi\downarrow & & \downarrow\phi \\ Y & \xrightarrow{g} & Y \end{array}$$
commutes

$\Rightarrow h(f) = h(g)$ with Bowen Def.

Proof: To start assume that ϕ is an isometry or $d'(\phi(x_1), \phi(x_2)) = d(x_1, x_2)$ then ϕ takes (n, ε)-separated sets for f on X to (n, ε)-separated sets for g on Y so $h(f) = h(g)$

Now in the general case define for $y_1, y_2 \in Y$

$$d''(y_1, y_2) = d(\phi^{-1}(y_1), \phi^{-1}(y_2))$$

<u>Claim</u>

(1) d'' is a metric on Y

(2) $\phi : (X, d) \to (Y, d'')$ is an isometry

(3) d' and d'' are equivalent metrics on Y

Given the claim, the special case and (2) imply $h(f) = h''(g)$ and by (3) and the Theorem on page 5 $h'(g) = h''(g)$ and so $h(f) = h'(g) = h(g)$

(the primes indicate the metric used to compute h)

Now to prove the claim

(1) is an exercise. For (2) note that

$d''(\phi(x_1), \phi(x_2)) = d(\phi^{-1}(\phi(x_1)), \phi^{-1}(\phi(x_2))) = d(x_1, x_2)$

For (3) we need to show

$\forall \varepsilon \exists \delta \; B_\delta(y, d'') \subseteq B_\varepsilon(y, d')$, $\forall \varepsilon \exists \delta' \; B_{\delta'}(y, d') \subseteq B_\varepsilon(y, d'')$

or $\forall \varepsilon \exists \delta \; d''(y_1, y_2) < \delta \Rightarrow d'(y_1, y_2) < \varepsilon$, $\forall \varepsilon \exists \delta' \; d'(y_1, y_2) < \delta' \Rightarrow d''(y_1, y_2) < \varepsilon$

or $\forall \varepsilon \exists \delta \; d(\phi^{-1}(y_1), \phi^{-1}(y_2)) < \delta \Rightarrow d'(y_1, y_2) < \varepsilon$

$\forall \varepsilon \exists \delta' \; d'(y_1, y_2) < \delta' \Rightarrow d(\phi^{-1}(y_1), \phi^{-1}(y_2)) < \varepsilon$

The last is the continuity of ϕ^{-1} and for the first let $x_1 = \phi^{-1}(y_1)$, $x_2 = \phi^{-1}(y_2)$ and it becomes the continuity of ϕ.

∎

Entropy in Shift spaces

- If $\Xi \subseteq \Sigma_n$ is compact and shift invariant $\sigma(\Xi) = \Xi$, then Ξ is called a subshift.

- The <u>language</u> of Ξ is the set of all finite words $b_0 b_1 \ldots b_{k-1}$ that occur in any sequence $\underline{s} \in \Xi$

- $W_n(\Xi) =$ number of words of length n in the language of Ξ

- The exponential complexity of the language of $\overline{X}$, or of $\overline{X}$, is

$$ec(\overline{X}) = \lim_{n\to\infty} \frac{\log w_n(\overline{X})}{n}$$ where we must prove the limit exists.

- Thus if $ec(\overline{X}) = \log(\eta) \Rightarrow w_n(\overline{X}) \sim \eta^n$

- FACT: $ec(\overline{X})$ exists and $ec(\overline{X}) = \inf_n \frac{\log w_n(\overline{X})}{n}$

Proof Since any allowable word of length $m+k$ must consist of an allowable word of length m followed by one of length k,

$$W_{m+k}(\overline{X}) \leq W_m(\overline{X})\, W_k(\overline{X})$$

and so $a_n = \log W_n(\overline{X})$ is subadditive and the result follows from the subadditive convergence Theorem.

It turns out that the exponential growth rate is the same as the topological entropy

<u>Theorem</u>: If $X \subseteq \Sigma_n$ is a subshift

$$h_{top}(\sigma) = ec(X).$$

For the proof we need a bit more information on the shift spaces. Σ_n.

Recall that the metric we are using (there are many equivalent ones) is

$$d(\underline{s}, \underline{t}) = \frac{1}{2^\ell} \quad \text{where } \ell = \min\{|i| : s_i \neq t_i\}$$