Entropy in Shift spaces

- If $\underline{X} \subseteq \Sigma_k$ is compact and shift invariant $\sigma(\underline{X}) = \underline{X}$, then $\underline{X}$ is called a subshift.

- The <u>language</u> of $\underline{X}$ is the set of all finite words $b_0 b_1 \ldots b_{\ell-1}$ that occur in any sequence $\underline{s} \in \underline{X}$

- $W_n(\underline{X}) =$ number of words of length n in the language of $\underline{X}$

- The exponential complexity of the language of X, or of X, is

$$ec(X) = \lim_{n\to\infty} \frac{\log W_n(X)}{n}$$ where we must prove the limit exists.

- Thus if $ec(X) = \log(m) \Rightarrow W_n(X) \sim m^n$.

- FACT: $ec(X)$ exists and $ec(X) = \inf \frac{\log W_n(X)}{n}$

<u>Proof</u> Since any allowable word of length $m+k$ must consist of an allowable word of length m followed by one of length k,

$$W_{m+k}(X) \leq W_m(X)\, W_k(X)$$

and so $a_n = \log W_n(X)$ is subadditive and the result follows from the subadditive convergence theorem.

It turns out that the exponential growth rate is the same as the topological entropy

<u>Theorem</u>: If $X \subseteq \Sigma_n$ is a subshift

$$h_{top}(\sigma) = ec(X).$$

For the proof we need a bit more information on the shift spaces. Σ_k.

Recall that the metric we are using (there are many equivalent ones) is

$$d(\underline{s}, \underline{t}) = 1/2^{\ell} \quad \text{where } \ell = \min\{|i| : s_i \neq s_j\}$$

<u>Proof:</u> Recall that S is (n,ε)-spanning if $\forall x\ \exists y \in S$ with $d_n(x,y) < \varepsilon$. Given the definition of the metric d on Σ_k we may assume $\varepsilon = \frac{1}{2^{\ell}}$ for some $\ell > 0$.

Let $T_1 = \{ [b_{-\ell} \dots b_0 \dots b_\ell] : b_{-\ell} \cdots b_\ell$ is an allowable word in X of length $2\ell+1\}$ and S_1 be the set created by picking one $\underline{s} \in X$ out of each cylinder set in T_1. Then any $\underline{t} \in X$ is within ε of a point in S_1 and no smaller collection has this property.

Thus $\mathcal{S}(1,\varepsilon) = \min\{\#S : S \text{ is } (1,\varepsilon)\text{-spanning}\}$

$= W_{2\ell+1}(X)$.

For $n \geq 1$, let

$$T_n = \{ [b_{-\ell} \ldots b_0 \ldots b_\ell \ldots b_{\ell+n}] : b_{-\ell} \ldots b_{\ell+n} \text{ is}$$

an allowable word in X of length $2\ell+1+n$. and build S_n by choosing one sequence from each of the cylinder sets in T_n. Now once again, any $\underline{x} \in X$ is within \epsilon some point in S_n in the d_n-metric and no shorter list of points has this property

Thus $S(n, \varepsilon) = W_{2\ell+1+n}(X)$

and using the definition of the topological entropy at scale ε,

$$h(\sigma, \varepsilon) = \lim_{n \to \infty} \frac{\log(S(n, \varepsilon))}{n}$$

$$= \lim_{n \to \infty} \frac{\log(W_{2\ell+1+n}(\Sigma_M))}{n}$$

$$= \lim_{n \to \infty} \frac{2\ell+1+n}{n} \frac{\log(W_{2\ell+1+n}(\Sigma_M))}{2\ell+1+n}$$

$$= \lim_{j \to \infty} \frac{\log(W_j(\Sigma_M))}{j} = ec(\Sigma_M)$$

Now note this is independent of ε and so

$$h(\sigma) = \lim_{\varepsilon \to 0} h(\sigma, \varepsilon) = ec(\Sigma_M). \quad \blacksquare$$

<u>Example</u>: Let $X = \Sigma_K$ then $W_n(X) = K^n$

and $h_{top}(\sigma) = \lim_{n\to\infty} \frac{\log K^n}{n} = \log K$

- The next result computes the topological entropy of a topological Markov chain.

Let M be a zero-one matrix and Σ_M be the corresponding topological Markov chain

$$\Sigma_M = \{ \underline{s} \in \Sigma_n : A_{s_i s_{i+1}} = 1 \ \forall i \}$$

<u>Theorem</u>: If M is irreducible then

$$h(\sigma|\Sigma_M) = \log \lambda \quad \text{where } \lambda > 0$$

is the largest eigenvalue in modulus.

Proof: Since $ec(\Sigma_M) = h(\sigma|\Sigma_M)$ by the previous result we need to compute $W_n(\Sigma_M)$

First a Lemma

Lemma: The number of allowable words in Σ_M of the form $i \cdots\cdots j$ with length k is

$$(M^{k-1})_{ij}$$

Proof: Let's start with an example with $n=3$, $k=2$

$$\begin{tikzcd}
 & 1 \arrow[dr, "M_{1j}"] & \\
i \arrow[ur, "M_{i1}"] \arrow[r, "M_{i2}"] \arrow[dr, "\cancel{m_{i3}}"'] & 2 \arrow[r, "M_{2j}"] & j \\
 & 3 \arrow[ur, "M_{3j}"'] &
\end{tikzcd}$$

If a path k has 1 then 1
ikj is allowable
if there is a zero, it is not

So # allowable words of length 3 starting with i and ending with j is $\displaystyle\sum_{k=1}^{3} M_{ik} M_{kj} = (M^2)_{ij}$

So In general, # available $i * j$ (* is wildcard) is

$$\sum_{w=1}^{n} M_{iw} \cdot M_{wj} = (M^2)_{ij}. \text{ Then}$$

by induction, $(M^{k-1})_{ij}$ is the number of allowable length k paths starting with i and ending with j ∎

Now we need another Lemma using the Perron-Fröbenius Theorem

Lemma If M is $n \times n$ and irreducible with largest eigenvalue $\lambda > 0$ and its eigenvector $\vec{v} > 0$

$\Rightarrow \exists\, c_1, c_2 > 0$ so that

$$c_1 \lambda^k \leq \sum_{\substack{i=1 \\ j=1}}^{n} (M^k)_{ij} \leq c_2 \lambda^k$$

$\forall k$.

<u>PROOF</u>: The eigenvalue-eigenvector pair $\lambda, \vec{v}$ as stated follow from Perron-Frobenius.

Let a and b be such that $0 < a < v_i < b$ $i = 1, \ldots, n$.

Since $M\vec{v} = \lambda\vec{v}$, $M^k \vec{v} = \lambda^k \vec{v}$ or

Fix i,

$$\sum_{j=1}^{n} (M^k)_{ij} v_j = \lambda^k v_i$$

Thus $a \sum_{j=1}^{n} (M^k)_{ij} \le \sum_{j=1}^{n} (M^k)_{ij} v_i = \lambda^k v_i \le b \sum_{j=1}^{n} (M^k)_{ij}$ (*)

Divide by a and sum over i yields

$$\sum_{i,j=1}^{n} (M^k)_{ij} \le \frac{1}{a} \sum_{i=1}^{n} \lambda^k v_i \le \frac{b}{a} \sum_{i=1}^{n} \lambda^k = \left(\frac{nb}{a}\right) \lambda^k$$

so $C_2 = \frac{nb}{a}$. A similar argument dividing (*) by (b) and summing yields $C_1 = \frac{na}{b}$ ∎

Proof that $h(\sigma|_{\Sigma_M}) = \log \lambda$

By the lemma on page 9

$$W_k(\Sigma_M) = \sum_{\substack{i=1 \\ j=1}}^{n} (M^k)_{ij}$$

and so if M is irreducible, the lemma on page 10 yields $\exists\, c_1, c_2 > 0$ with

$$c_1 \lambda^k \leq W_k(\Sigma_M) \leq c_2 \lambda^k$$

so

$$\frac{\log c_1 + k \log \lambda}{k} \leq \frac{\log W_k(\Sigma_M)}{k} \leq \frac{\log c_2 + k \log \lambda}{k}$$

letting $k \to \infty$ ~~yie~~ with the theorem on page 3 yields $ec(\Sigma_M) = h_{top}(\sigma|_{\Sigma_M}) = \log \lambda$ ∎

<u>Remark</u> $\log(\lambda)$ is also the measure theoretic entropy of the Parry measure on Σ_M.

<u>Example</u> $M = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \lambda = \frac{1 \pm \sqrt{1^2+4}}{2}$

so $h_{top}(\sigma|\Sigma_M) = \log\left(\frac{1+\sqrt{5}}{2}\right)$.