MISC RESULTS

- All the constructions and results for two-sided shifts like
  - Construction of Parry measure
  - entropy of Parry measure
  - Topological entropy = exponential complexity
  - Topological entropy of topological Markov chains
  - etc

All hold with the same formulas for one-sided subshifts $\subseteq \Sigma_k^+ = \{ .s_0 s_1 \dots \in \{0, \dots, k-1\}^{\mathbb{N}} \}$

Another result on topological entropy

<u>Theorem</u>: If X is compact, metric $f: X \to X$ continuous.

(1) $h(f^m) = m\,h(f)$ for $m \geq 0$

(2) If f is a homeomorphism then $h(f^{-1}) = h(f)$ and so $h(f^m) = |m|\, h(f)$ $\forall m \in \mathbb{Z}$

where all $h = h_{top}$.

Proof (1) Since

$$\max_{0 \leq i \leq n} d(f^{mi}(x), f^{mi}(y)) \leq \max_{0 \leq j \leq mn} d(f^j(x), f^j(y))$$

we have $S(n, \varepsilon, f^m) \leq S(mn, \varepsilon, f)$ where we now include the function in the spanning number. Passing to the limits we get $h(f^m) \leq m\,h(f)$

Conversely, for an $\varepsilon > 0$ there is a $\delta(\varepsilon) > 0$ so that $d(x,y) < \delta(\varepsilon) \Rightarrow d(f^i(x), f^i(y)) < \varepsilon$ for $i = 0, \dots, m$. Thus

$$S(n, \delta(\varepsilon), f^m) \geq S(mn, \varepsilon, f) \text{ and}$$

Since $\varepsilon \to 0 \Leftrightarrow \delta(\varepsilon) \to 0$ passing to the limits:

$h(f^m) \geq m\, h(f)$.

(2) The $n\underline{\Omega}$-image of a (n,ε)-separated set for f^{-1} and f is a (n,ε)-separated set for f and vice versa. ∎

<u>NOTE</u>: Both these results are also true for measure theoretic entropy for mpt and bi-mpt.

Semi conjugacies

Thm $f: \mathbb{X} \to \mathbb{X}$, $g: \mathcal{Y} \to \mathcal{Y}$ are continuous, onto and $\alpha: \mathbb{X} \to \mathcal{Y}$ is continuous, onto and

$$\begin{array}{ccc} \mathbb{X} & \xrightarrow{f} & \mathbb{X} \\ {\scriptstyle \alpha}\downarrow & & \downarrow{\scriptstyle \alpha} \\ \mathcal{Y} & \xrightarrow{g} & \mathcal{Y} \end{array}$$

commutes (This says $(\mathbb{X}, f)$ is semi-conjugate to $(\mathcal{Y}, g)$)

$$\Rightarrow \quad h_{top}(f) \geq h_{top}(g).$$

We acturally proved this as part of the result when α is a homeomorphism.

Remark: The analogous result is true for measure theoretic entropy.

The two definitions of topological entropy are the same.

<u>Theorem</u> $f: X \to X$ is continuous map of compact metric spaces then $h_{top}(f)$ as computed with the open set definition and Bowen's definition give the same result.

<u>Main lemma in proof</u>: If α is an open cover of X by open balls of radius 2ε and γ is an open cover by open balls of radius $\varepsilon/2$ then

$$N\left(\bigvee_{i=0}^{n-1} f^{-i}\alpha\right) \le S(n,\varepsilon) \le N\left(\bigvee_{i=0}^{n-1} f^{-i}\gamma\right)$$

then take logs and divide by n pass to limit . . . .

The variational principle

<u>Theorem</u>: $f : X \to X$ is a continuous function of a compact metric space then

$$h_{top}(f) = \sup\{ h_\mu(f) : \mu \text{ is an invariant Borel probability measure} \}$$

<u>Remark</u> Recall that we showed that $spt(\mu) \subseteq \Omega(f)$ for μ an invariant, Borel prob. and $\Omega(f)$ the non wandering set. When f is a homeomorphism, $f(\Omega(f)) = \Omega(f)$ and so

h_{top}

$$h_{top}(f) = h|_{\Omega(f)}$$

so all the entropy is carried by the non-wandering set

If $h_\mu(f) = h_{top}(f) \Rightarrow \mu$ is called a measure of maximal entropy.

Theorem: Under the same hypothesis, if $h_{top}(f) < \infty$ then there are measures of maximal entropy (there could be one or many).

Example: If Σ_M is a topological Markov chain with M irreducible, then the Parry measure is the unique measure of maximal entropy.

Flows

Ergodic theory had its origins not in iterated maps but rather in flows, or solutions to differential equations.

(i). $x \in \mathbb{R}^n$

$$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}) \quad \text{is well-behaved}$$

Then for each x_0 $\exists$ $\vec{x}(t;\vec{x}_0)$ with

$$\frac{d\vec{x}(t;\vec{x}_0)}{dt} = F(\vec{x}(t;\vec{x}_0))$$

These solutions are collected together in

$$\varphi : \mathbb{R} \times X \to X$$

$$\varphi(t, x_0) = \vec{x}(t; x_0)$$

so $$\frac{\partial \varphi}{\partial t} = \vec{F}(\varphi)$$

This is usually written $\varphi_t(x)$

It turns out that such a φ satisfies the group property

$$\varphi_0 = \mathrm{Id}, \qquad \varphi_t \circ \varphi_s = \varphi_{t+s} \qquad (*)$$

which says that $\varphi : \mathbb{R} \times X \to X$ is an action of $\mathbb{R}$ on X

DEF: If $\varphi : \mathbb{R} \times X \to X$ satisfies $(*)$ and

(a) φ is differentiable $\Rightarrow$ φ is a smooth flow

(b) φ is continuous $\Rightarrow$ φ is a continuous flow

(c) φ is measurable (so X has a measure μ and $\mathbb{R}$ has Lebesgue $\Rightarrow$ φ is a measurable flow

DEF: If μ is a prob measure on $\underline{X}$

$\Rightarrow \varphi$ is a measure preserving flow on $\underline{X}$ (mpf)

if $(\varphi_t)_* \mu = \mu \quad \forall t \in \mathbb{R}$ i.e.

$$\mu((\varphi_t)^{-1} A) = \mu(A) \quad \forall A \in \mathcal{B} = \text{sigma alg of } \mu$$

RK: note that from the group property (*)

$(\varphi_t)^{-1} = \varphi_{-t}$, so we can also write

$$\mu(\varphi_t(A)) = \mu(A) \quad \forall t.$$

DEF φ_t is a mpf and is ergodic if

$$\varphi_t(A) = A \ \forall t \Rightarrow \mu(A) = 0 \text{ or } \mu(A) = 1$$

Ergodic Theorem for flows φ is a mpf on $(X, \mu, \mathcal{B})$. φ is ergodic $\Longleftrightarrow$ $\forall \alpha \in L^1(\mu)$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T \alpha\, \varphi_t(x) \to \int_X \alpha \, d\mu \quad \text{a.e. } x.$$

- Mixing is also defined, ..., isomorphism,
- The entropy of φ is the entropy of φ_1, the time one map. Note that this depends on parameterization of the flow.