

MISC RESULTS

• All the constructions and results for

two-sided shifts like

- Construction of Parry measure
- entropy of Parry measure = exponential complexity
- Topological entropy of topological Markov chains
- etc

formulas for one-sided

All hold with the same formulas for $\{9, 10, 11, 12, 13, 14, 15\}$

$$\sum_k^+ = \sum \cdot s_0 s_1 \dots$$

sub shifts \subseteq

Another result on Topological entropy

Theorem: If X is compact, metric $f: X \rightarrow X$ continuous.

- (1) $h(f^m) = m h(f)$ for $m \geq 0$ then
- (2) If f is a homeomorphism $\forall m \in \mathbb{Z}$
- $h(f^{-1}) = h(f)$ and so $h(f^m) = |m| h(f)$ $\forall m \in \mathbb{Z}$

where all $h = h_{top}$.

Proof (i) Since $d(f^{m_1}(x), f^{m_2}(y)) \leq \max_{0 \leq j \leq mn} d(f^j(x), f^j(y))$ where we

have $\Delta(n, \epsilon, f^m) \leq \delta(mn, \epsilon, f)$ where δ is a spanning number. Posing n to be ϵ we get $h(f^m) \leq m h(f)$

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conversely, for an $\epsilon > 0$ there is a $\delta(\epsilon) > 0$
so that $d(x, y) < \delta(\epsilon) \Rightarrow d(f^i(x), f^i(y)) < \epsilon$

for $i = 0, \dots, m$. Thus

$$S(n, \delta(\epsilon), f^m) \geq S(mn, \epsilon, f) \text{ and}$$

Since $\epsilon \rightarrow 0 \Leftrightarrow \delta(\epsilon) \rightarrow 0$ passing to the limits.

$$h(f^m) \geq m h(f).$$

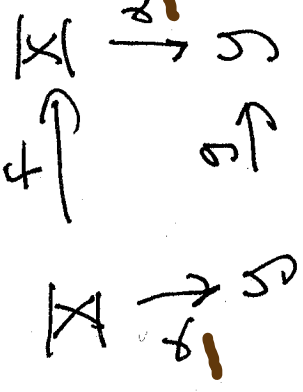
(2) The $n\mathbb{Z}$ -image of a (n, ϵ) -separated set for f is a (n, ϵ) -separated set for f^{-1} and vice versa. \blacksquare

NOTE: Both these results are also true for measure theoretic entropy for mpt and bi-mpt.

Semi conjugacies

$f: X \rightarrow Y, g: Y \rightarrow X$ are continuous, onto

$\alpha: X \rightarrow Y$ is continuous, onto and



commutes

(This says (X, f) is semi-conjugate to (Y, g))

$$\Rightarrow h_{top}(f) \geq h_{top}(g)$$

We actually proved this as part of the result when α is a homeomorphism.

Remark: The analogous result is true for measure theoretic entropy.

The two definitions of topological

Entropy are the same.

Theorem $f: X \rightarrow Y$ is continuous map of compact metric spaces then $h_{top}(f)$ as computed with the open set definition and Bowen's definition

give the same result.

Main Lemma in proof: If α is an open cover of X by open balls of radius $\frac{\epsilon}{2}$ and δ is an open cover by open balls of radius $\frac{\epsilon}{2}$

then take logs and divide by n and take limit $n \rightarrow \infty$

$$N(\bigvee_{i=0}^{n-1} f^{-i} \alpha) \leq S(n, \epsilon) \leq \mu \left(\bigvee_{i=0}^{n-1} f^{-i} \delta \right)$$

then take logs

and divide by n and take limit $n \rightarrow \infty$

The variational principle

function

Theorem: $f: X \rightarrow X$ is a continuous map of a compact metric space then

$$h_{\text{top}}(f) = \sup \{ h_{\mu}(f) : \mu \text{ is an invariant Borel probability measure.} \}$$

that $\text{spt}(\mu) \subseteq \Omega(f)$

Remark Recall that we showed for μ an invariant, Borel prob. and $\Omega(f)$ is a non-wandering set. When f is a homeomorphism,



$$h(\Omega(f)) = h_{\text{top}}(f)$$

$$h_{\text{top}}(f) = h(\Omega(f))$$

so all the entropy is carried by the non-wandering set

If $h_\mu(f) = h_{\text{top}}(f) \Rightarrow \mu$ is called a measure

of maximal entropy.

Theorem: Under the same hypothesis, if $h_{\text{top}}(f) < \infty$ then there are one or many.

(There could be one or many).

Example If Σ_μ is a topological Markov chain

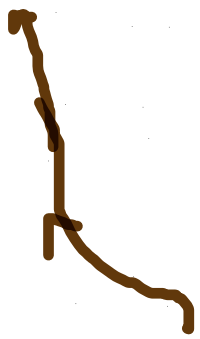
with M irreducible, then the Parry measure is the unique measure of maximal entropy.

Flows

had its origins not in targeted ergodic theory flows, or solutions to maps but rather in differential equations.

Q1. $x \in \mathbb{R}^n$

$\frac{d\vec{x}}{dt} = \vec{F}(\vec{x})$ is well-behaved



Then for each $x_0 \exists \vec{x}(t; \vec{x}_0)$ with

$$\frac{d\vec{x}(t; \vec{x}_0)}{dt} = \vec{F}(\vec{x}(t; \vec{x}_0))$$

These solutions are collected together in

$$\boxed{\varphi} : \mathbb{R} \times \mathbb{R}^n \Rightarrow \mathbb{R}^n$$
$$\varphi(t, x_0) = \vec{x}(t; x_0)$$

so $\frac{\partial \varphi}{\partial t} = \vec{F}(\varphi)$

This is usually written $\varphi_t(x)$

It turns out that such a φ satisfies the

group property

$$\varphi_0 = \text{id} \quad (*)$$

$$\varphi_s \circ \varphi_t = \varphi_{t+s}$$



which says that $\varphi: \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{X}$ is an action of \mathbb{R} on \mathbb{X}

DEF: If $\varphi: \mathbb{R} \times \mathbb{X} \rightarrow \mathbb{X}$ satisfies $(*)$ and

- (a) φ is differentiable $\Rightarrow \varphi$ is a smooth flow
- (b) φ is continuous $\Rightarrow \varphi$ is a continuous flow
- (c) φ is measurable (so \mathbb{X} has a measure μ) and \mathbb{R} has Lebesgue $\Rightarrow \varphi$ is a measurable flow

DEF: If μ is a prob measure on X

(mpf)

flow on X

$\Rightarrow \varphi$ is a measure preserving

ie.

$$\text{if } (\varphi_t)_* \mu = \mu \quad \forall t \in \mathbb{R}$$

$$\forall A \in \mathcal{B} = \text{sigma alg of } \mu$$

$$\mu(\varphi_t^{-1}(A)) = \mu(A)$$

from the group property (φ_t)

we can also write

$$(\varphi_t^{-1})_* \mu = \mu$$

$\forall t$.

$$\mu(\varphi_t(A)) = \mu(A)$$

φ_t is a mpf and is ergodic if

DEF

$$\varphi_t(A) = A \quad \forall t \Rightarrow \mu(A) = 0 \text{ or } \mu(A) = 1$$

Ergodic Theorem for flows

(X, \mathcal{M}, μ). φ is ergodic $\Leftrightarrow \forall A \in \mathcal{M}$ $\mu(A) = 0$ or $\mu(A) = 1$ $\Leftrightarrow \forall \alpha \in \mathbb{R}$ $\mu(\varphi_\alpha^{-1} A) = \mu(A)$

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \alpha \varphi_t(x) dx = \int_X \alpha d\mu \quad \text{a.e. } x$$

- Mixing is also defined as, isomorphism.
- The entropy of φ is the entropy of φ_1 . Note that this depends on the time one map.
- Parameterization of the flow.