Product measures and shift spaces

- Another important application of the extension theorems is to the construction of product space topologies
- We first do the two-sided product. For each $i \in \mathbb{Z}$ $(X_i, \mathcal{B}_i, \mu_i)$ is a probability space. For $-\infty < m \leq n < \infty$ consider for $A_i \in \mathcal{B}_i$ $m \leq i \leq n$

$$R = \prod_{i=-\infty}^{m-1} X_i \times \prod_{i=m}^{n} A_i \times \prod_{i=n+1}^{\infty} X_i$$

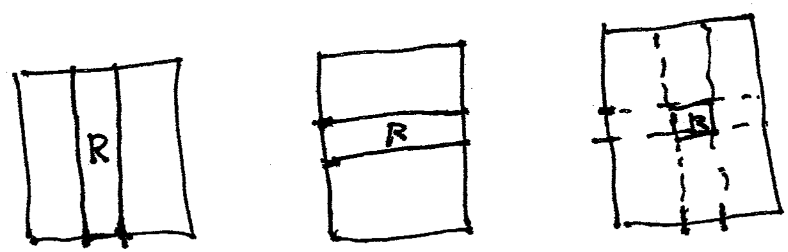
- This is called a measurable rectangle and the collection of finite disjoint unions of these is a Algebra
- $(X, \mathcal{B}) = \prod_{i=-\infty}^{\infty} (X_i, \mathcal{B}_i)$ is the notation for the σ-algebra generated by the Algebra and is called the product σ-algebra

E

(2)

- Let $I(R) = \prod_{i=m}^{n} \mu_i(A_i)$ and extend I to a probability measure μ on $\mathcal{B}$ called the product measure. (or more properly, the direct product)

- $[0,1] \times [0,1]$



If we are given $(X_i, \mathcal{B}_i, \mu_i)$ for $\underline{\underline{i \in \mathbb{N}}}$ we have rectangles for $0 \le m \le n < \infty$

$$R = \prod_{i=0}^{m-1} X_i \times \prod_{i=m}^{n} A_i \times \prod_{i=n+1}^{\infty} X_i$$

and proceed as with $i \in \mathbb{Z}$.

- A special, very important case is when all $X_i = \{0, 1, .., n-1\}$ for some n. In this case we only need consider A_i = a single point. We also put the same measure on each X_i which is thus a probability vector $\vec{p} = (p_0, p_1, .., p_{n-1})$ with $p_i \geq 0$ and $\sum p_i = 1$

measure of k is p_k

- The rectangles are now called <u>cylinder sets</u> and determined by a <u>block</u> $b_0 b_1 .. b_k$ with $b_i \in \{0, 1, .., n-1\}$ and the starting point

and $_m[b_0 b_1 .. b_k] = \{ \underline{x} \in \prod_{i=-\infty}^{\infty} \{0, 1, .., n-1\} : x_{m+i} = b_i$ for $i = 0, .., k\}$

- If the block is called $B = (b_0 b_1 .. b_k)$ then this is written $_m[B]$

- The product measure is $\mu\left(_m[b_0 .. b_k]\right) = \prod_{i=0}^{k} p_{b_i}$ ← lower case p.

- A similar construction on $\prod_{i=0}^{\infty} \{0, 1, .., n-1\}$

<u>Notation</u>: $\prod_{i=-\infty}^{\infty} \{0,\dots,n-1\} = \Sigma_n = \{0,\dots,n-1\}^{\mathbb{Z}}$

is full shift on n symbols and $\prod_{i=0}^{\infty} \{0,\dots,n-1\} = \Sigma_n^+ =$

$\{0,\dots,n-1\}^{\mathbb{N}}$ is the one sided shift on n symbols

or briefly, full n-shift and one-sided n-shift (the reason for "shift" we be clear shortly)

The measure $\mu_{\vec{p}}$ constructed from $\vec{p} = (p_0,\dots,p_{n-1})$

is called a Bernoulli measure.

<u>Example</u>: Σ_2 is all two sided sequences of 0's and 1's, $\underline{x} \in \Sigma_2$ is thus, for example, ..

$$\underline{x} = \dots 011.011001\dots,$$

$\uparrow$ x_0

in general, $\underline{x} = \dots x_{-2}x_{-1}.x_0x_1x_2\dots$ $\quad x_i \in \{0,1\}$

$C = {}_3[10011]$ a cylinderset, so $\underline{x} \in C$

looks like a sequence

$$\ldots **.**10011**\ldots$$

where $*$ can be a zero or a one.

If $\vec{p} = \left(\frac{1}{4}, \frac{3}{4}\right) \Rightarrow \mu_{\vec{p}}(C) = \frac{3}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{3}{4} \cdot \frac{3}{4}$

Now let $\vec{p} = (1/2, 1/2)$ and conside $\mu_{\vec{p}}$ on Σ_2^+

so $\underline{x} \in \Sigma_2^+$ looks like $.0100\ldots$, for example.

This models an infinite series of coin tosses by a fair coin $0 = H$, $1 = T$ and now $\mu_{\vec{p}}(C) = (1/2)^5$ is the probability of the 3rd toss being tails, the 4th heads, the 5th heads, the 6th tails and the 7th tails.

The shift transformation

- Define $\sigma : \Sigma_n \to \Sigma_n$ via $\sigma(\ldots x_{-2} x_{-1} x_0 \bullet x_1 x_2 \ldots)$ $= \ldots x_{-2} x_{-1} x_0 \bullet x_1 x_2 \ldots$ so $\sigma(\underline{x})_i = x_{i+1}$

  So it is easy to see that σ is a bijection

- Define $\sigma : \Sigma_n^+ \to \Sigma_n^+$ via $\sigma(\bullet x_0 x_1 x_2 \ldots)$ $= \bullet x_1 x_2 x_3 \ldots$. So this σ is n to 1.

- These transformations are called the "shift maps" (or more properly, the left shift)

- And again more properly, the pair (Σ_n, σ) is the full n-shift and (Σ_n^+, σ) the one sided n-shift

<u>Proposition</u> A Bernoulli measure on Σ_n^+ or Σ_n is shift invariant

<u>Proof</u> $\sigma^{-1}({}_k[B]) = {}_{k+1}[B]$ so

$$\mu_{\vec{p}}(\sigma^{-1}({}_k[B])) = \mu_{\vec{p}}({}_{k+1}[B]) = \mu_{\vec{p}}({}_k[B])$$

it is invariant on the generating semialgebra $\Rightarrow$ invariant $\blacksquare$

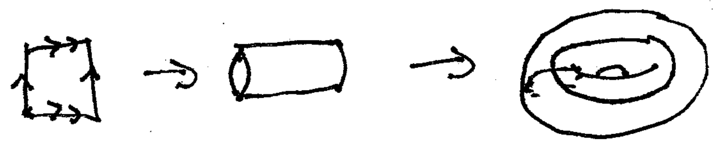
Using Theorem 6 from last lecture

<u>Interpretation</u> in coin toss: the probability of a sequence HTTH in any 4 sequential tosses is the same.

Examples on the 2-dimensional torus

- Let $\mathbb{T}^2 = S^1 \times S^1$ and let μ be the product measure on $\mathbb{T}^2$ coming from Lebesgue measure on S^1.

- one can check this is the same measure we get from Lebesgue measure on $[0,1]^2$ and then identify edges to get $\mathbb{T}^2$



- We also treat $\mathbb{T}^2$ as $\mathbb{R}^2 / \mathbb{Z}^2$ so

$$(x,y) \sim (x',y') \iff x' = x+m \quad y' = y+n, \quad (m,n) \in \mathbb{Z}^2$$

- So $F : \mathbb{R}^2 \to \mathbb{R}^2$ descends to a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ iff $\forall (m,n) \in \mathbb{Z}^2 \ \exists (m',n') \in \mathbb{Z}^2$ with $F(x+m, y+n) = F(x,y) + (m',n')$

$$\begin{array}{ccc} \mathbb{R}^2 & \xrightarrow{F} & \mathbb{R}^2 \\ \downarrow & & \downarrow \\ \mathbb{T}^2 & \xrightarrow{f} & \mathbb{T}^2 \end{array}$$

(1) Let $F(x,y) = (x+\alpha_1, y+\alpha_2)$ for $\vec{\alpha} \in \mathbb{R}^2$, then

F descends to $f: \mathbb{T}^2 \to \mathbb{T}^2$. Now the derivative

$DF(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ so $\det DF \equiv 1$, so F preserves

Lebesque on $\mathbb{R}^2$ and f preserves μ on $\mathbb{T}^2$

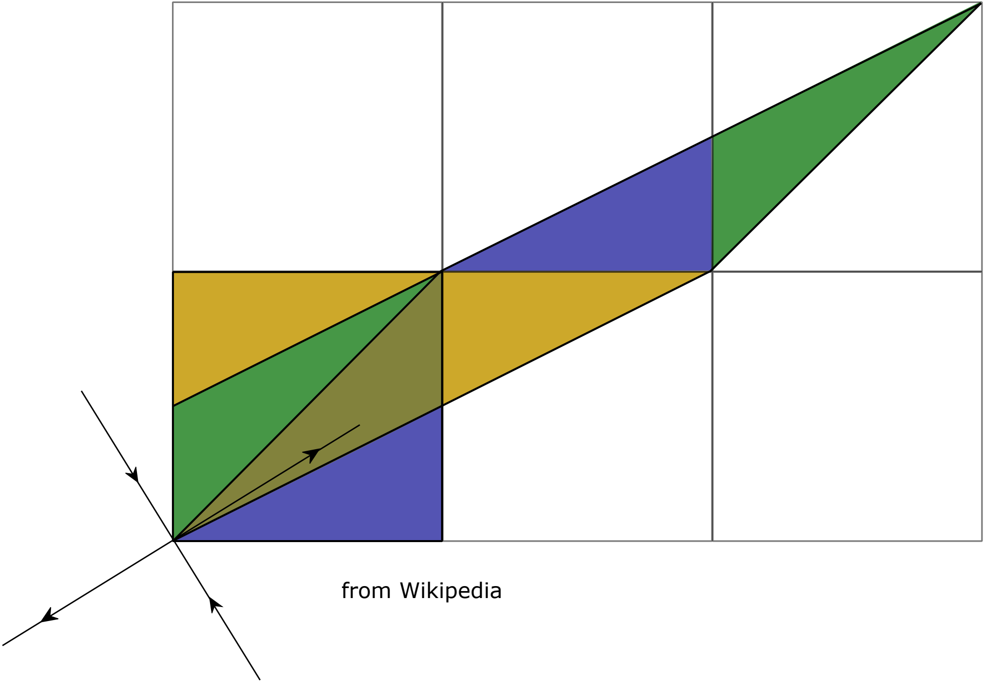
(2) $F(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$ then F also descends

and $\det DF(x,y) = \det \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 1$ so f also

preserves μ

This example (2) is the famous "cat map" of

Arnol'd or Thom's toral automorphism.



from Wikipedia

Ergodicity

- As with many areas of mathematics one searches for the indecomposable objects in the category of mpt.

- Recall in topological dynamics the strongest notion of indecomposibility is minimality: $f : X \to X$ contains no non-trivial compact subinvariant sets

- In ergodic theory, a mpt is ergodic if it contains no non-trivial (in the sense of measure) subinvariant sets

(NOTE: We always assume $f: X \to X$ is onto unless otherwise stated - I should have said that earlier)

DEF: Assume $f:(X, \mathcal{B}, \mu) \circlearrowleft$ is a mpt. It is *ergodic* if for $A \in \mathcal{B}$, $f^{-1}(A) = A$ implies either $\mu(A) = 0$ or $\mu(A) = 1$

Remark If $f^{-1}(A) = A \Rightarrow A = ff^{-1}(A) = f(A)$ so we consider $f|_A$. If $0 < \mu(A) < 1$ we then have a nontrivial sub-mpt.

Examples (to be proved) - These are ergodic

(1) $R_\alpha : S^1 \to S^1$ when $\alpha \notin \mathbb{Q}$ with Leb meas.

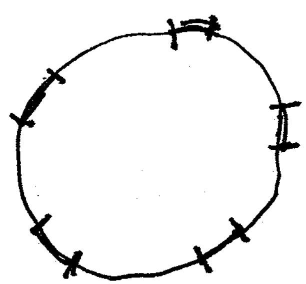
(2) $\sigma : \Sigma_n \to \Sigma_n$ with $\mu_{\vec{p}}$ an Bernoulli measure with all $p_i > 0$

(3) $f: S^1 \to S^1$ $f(\theta) = n\theta$ $n > 1$ with Leb.

(4) $f: \mathbb{T}^2 \circlearrowleft$ $f(x, y) = (x + \alpha_1, y + \alpha_2) \bmod \mathbb{Z}^2$ when α_1 and α_2 are rationally independent and Leb meas

(5) $f: \mathbb{T}^2 \circlearrowleft$ $f(x,y) = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \bmod \mathbb{Z}^2$ with Leb meas.

$R_{p/q}: S^1 \circlearrowleft$ is not ergodic



Lots of sub-invariant sets with measures strictly between 0 and 1.