<u>Th'm</u>: Let $\mu_{\vec{p}}$ be a Bernoulli measure on Σ_n determined by $\vec{p} = (p_0 \ldots p_{n-1})$. Then $\vec{\mu}_p$ is ergodic under σ. We abbreviate $\mu = \mu_p$ and recall $\mathcal{B}$ = Borels.

<u>Proof</u>: Recall that the collection of finite unions of cylinder sets is an algebra that generates the Borels.

Let $E \in \mathcal{B}$ be such that $\sigma^{-1}(E) = E$. We must show $\mu(E) = 0$ or 1.

Using the approximation Theorem of last lecture find a finite union of cylinder sets A with

$\mu(E \Delta A) < \varepsilon$ for a given ε. By (e),

$|\mu(E) - \mu(A)| < \varepsilon$ also.

- Now since A is the finite union of cylinder sets as in the sketch proof we may find N large enough so that A and $\sigma^{-N}(A)$ restrict different coordinates
- Thus letting $B = \sigma^{-N}(A)$ using the definition of μ as a product measure, $\mu(B \cap A) = \mu(B)\mu(A)$
$= \mu(A)^2$ since μ is σ-invariant.
- Now note $\mu(E \Delta B) = \mu(E \Delta \sigma^{-N}(A))$
$= \mu(\sigma^{-N}(E) \Delta \sigma^{-N}(A)) = \mu(\sigma^{-N}(E \Delta A)) = \mu(E \Delta A) < \varepsilon$
using the fact that σ is a homeomorphism.
using another basic fact $E \Delta (A \cap B) \subseteq (E \Delta A) \cup (E \Delta B)$
- we have $\mu(E \Delta (A \cap B)) \leq \mu(E \Delta A) \vee \mu(E \Delta B) < 2\varepsilon$

Finally

$$|\mu(E) - \mu(E)^2|$$

$$\le |\mu(E) - \mu(A \wedge B)| + |\mu(A \wedge B) - \mu(E)^2|$$

$\boxed{\mu(A \wedge B) = \mu(A)^2}$

$$\le 2\varepsilon + |\mu(A)^2 - \mu(E)^2|$$

$$\le 2\varepsilon + \mu(A)\,|\mu(A) - \mu(E)| + \mu(E)\,|\mu(A) - \mu(E)|$$

$$\le 2\varepsilon + 2\varepsilon = 4\varepsilon$$ using $\mu(A), \mu(E) \le 1$ and $|\mu(A) - \mu(E)| < \varepsilon$

Since ε is arbitrary, $\mu(E) = (\mu(E))^2$ and so $\mu(E) = 0$ or 1 $\blacksquare$

NOTE: This clever proof as with most of the proofs come from Walters' book.

- A similar proof yields the ergodicity of a Bernoulli measure on the one-sided shift.

Integration and L^p spaces

- Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $\alpha : X \to \mathbb{C}$ measurable (we sometimes consider just real valued or positive valued α)

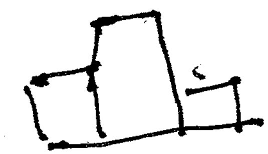
- For $A \in \mathcal{B}$, χ_A is its indicator function

$$\chi_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

- $\beta : X \to \mathbb{R}$ is a simple function if

$$\beta = \sum_{i=1}^{n} b_i \chi_{A_i}$$ with $A_i \in \mathcal{B}$ pairwise disjoint, $b_i \in \mathbb{R}$

- For a simple function β, define



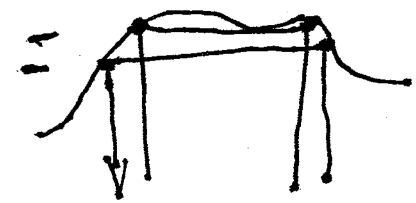
$$\int \beta \, d\mu = \sum_{i=1}^{n} b_i \, \mu(A_i)$$

- Now assume $\cancel{\beta} : \mathbb{X} \to \mathbb{R}^+$ is measurable (note it is ≥ 0)

  Then $\exists$ sequence of simple functions $\beta_n \uparrow \alpha$

  eg $\beta_n = \frac{i-1}{2^n}$, if $\frac{i-1}{2^n} \leq \alpha(x) < \frac{i}{2^n}$ $\quad i = 1, \dots, n2^n$

  $\quad = n$, if $\alpha(x) \geq n$



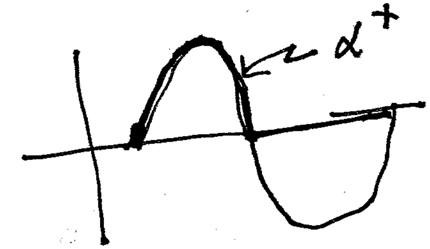
- Define $\int \alpha \, d\mu = \lim_{n \to \infty} \int \beta_n \, d\mu$

- This is independent of the choice of $\beta_n \uparrow \alpha$

- Say α is <u>integrable</u> if $\int \alpha \, d\mu < \infty$

- Now if $\alpha : X \to \mathbb{R}$ is measurable we may write $\alpha = \alpha^+ - \alpha^-$ where

$$\alpha^+(x) = \max\{\alpha(x), 0\} \geq 0$$
$$\alpha^-(x) = \max(-\alpha(x), 0) \geq 0$$



- α is <u>integrable</u> [if] both α^+ and α^- are and then define

$$\int \alpha \, d\mu = \int \alpha^+ d\mu - \int \alpha^- d\mu$$

- $\alpha : X \to \mathbb{C}$ is integrable if $\alpha = \alpha_1 + i\alpha_2$ and α_1 and α_2 are and then define

$$\int \alpha \, d\mu = \int \alpha_1 \, d\mu + i \int \alpha_2 \, d\mu$$

- If the domain needs to be emphasized: $\int_X \alpha \, d\mu$.

- Say $\alpha = \beta$ a.e. (almost everywhere) if

$$\mu(\{x : \alpha(x) \neq \beta(x)\}) = 0$$

- If $\alpha = \beta$ a.e., α is integrable $\Leftrightarrow$ β is and

$$\int \alpha \, d\mu = \int \beta \, d\mu$$

Function Spaces

- The collection of all measurable $\alpha : X \to \mathbb{C}$ with $|\alpha|^p$ integrable $(p \geq 1)$ is a vector space.
- Let $L^p(X, \mathcal{B}, \mu)$ $(= L^p(\mu)$ for short$)$ denote the vector space of equivalence classes.

where \alpha ~ \beta iff \alpha = \beta a.e.

- NOTE: it is usual to write $\alpha \in L^p(\mu)$ and ignore the fact that elements of $L^p(\mu)$ are actually equivalence classes, not functions

- For $\alpha \in L^p(\mu)$, let $\|\alpha\|_p = \left(\int |\alpha|^p \, d\mu\right)^{1/p}$ $p \geq 1$

- This yields a complete norm on $L^p(\mu)$ which makes it a Banach space

- The metric on $L^p(\mu)$ is $d_p(\alpha, \beta) = \|\alpha - \beta\|_p$ making it a complete, separable metric space.

- We now have the second theorem giving equivalent conditions to ergodicity

Theorem: $f : (X, \mathcal{B}, \mu)$ ⟲ is a mpt and $\alpha : X \to \mathbb{R}$ is measurable. The following are equivalent

(i) f is ergodic

(ii) $\alpha \circ f = \alpha$ implies α constant a.e.

(iii) $\alpha \circ f = \alpha$ a.e. implies α constant a.e.

(iv) $\alpha \in L^2$, $\alpha \circ f = \alpha \Rightarrow \alpha$ constant a.e.

(v) $\alpha \in L^2$, $\alpha \circ f = \alpha$ a.e. $\Rightarrow \alpha$ constant a.e.

Remark They all mean roughly "any invariant function is essentially constant"

Proof The implications shown by arrows are trivial

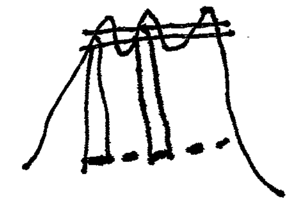
i
ii
iii
iv
v

Leaving only (i) $\Rightarrow$ (iii) and (iv) $\Rightarrow$ (i) to prove

(i) $\Rightarrow$ (iii) So assume f ergodic, α measurable
$\alpha \circ f = \alpha$ a.e.

For all $k \in \mathbb{Z}$ and $n > 0$ let

$$\overline{X}(k,n) = \{x;\ \frac{k}{2^n} \leq \alpha(x) < \frac{k+1}{2^n}\}.$$



$$= \alpha^{-1}\left(\left[\frac{k}{2^n}, \frac{k+1}{2^n}\right)\right)$$

Now $f^{-1}(\overline{X}(k,n)) \,\Delta\, \overline{X}(k,n) \subseteq \{x;\ \alpha \circ f(x) \neq \alpha(x)\}$.

and thus $\mu(f^{-1}(\overline{X}(k,n)) \,\Delta\, \overline{X}(k,n)) = 0$

Thus by the alternative definition of ergodicity proved in the last theorem

$$\mu(\overline{X}(k,n)) = 0 \text{ or } 1$$

Now fix n. Note that

$$X = \bigcup_{k\in\mathbb{Z}} \Xi(k,n)$$ is a disjoint union

of sets with measure 0 or 1. Thus for exactly one k_n, $\mu(\Xi_{k_n}, n) = 1$ and so

$$A = \bigcap_{n=1}^{\infty} \Xi(k_n, n)$$ has $\mu(A) = 1$

But α is constant on A, so α is constant a.e.

(iv) $\Rightarrow$ (i) Assume $E \in \mathcal{B}$, $f^{-1}(E) = E$

Now certainly $\chi_E \in L^2(\mu)$ and $f^{-1}(E) = E$ says exactly that $\chi_E \circ f = \chi_E$. Thus by (iv), χ_E is constant a.e. But it only takes on 2 values 0 or 1

So either $\chi_E = 0$ a.e. or $\chi_E = 1$ a.e. $\Rightarrow \mu(E) = \int \chi_E \, d\mu = 0$ or 1.

Remark: For conditions (iv) and (v) all that was needed in the proof was that $\chi_E \in L^2(\mu)$.

So it could also be stated for $\alpha \in L^p(\mu)$ for any chosen $p > 1$ or even α is a bounded function

We stated the theorem for L^2 as that is most useful for proving ergodicity on tori (next time).