$L^2(\mu)$

- $L^2(\mathbb{X}, \mathcal{B}, \mu)$ has additional structure which is quite useful. We now allow $\alpha \in L^2(\mu)$ to be complex valued
- For $\alpha, \beta \in L^2(\mu)$, define $\langle \alpha, \beta \rangle = \int \alpha \bar{\beta} \, d\mu$ where $\bar{\beta}$ is the complex conjugate of β
- $\langle \alpha, \beta \rangle$ is a complex inner product

$$\langle \beta, \alpha \rangle = \overline{\langle \alpha, \beta \rangle}$$

$$\langle c\beta, \alpha \rangle = c \langle \beta, \alpha \rangle$$

$$\langle \beta, c\alpha \rangle = \bar{c} \langle \beta, \alpha \rangle$$

and most importantly, $\|\alpha\|_2 = \langle \alpha, \alpha \rangle^{1/2}$.

- This makes $L^2(\mu)$ a <u>Hilbert space</u>

- If X is a metric space and $\mathcal{B}$ be Borels, and μ a finite measure

then $L^2(X, \mathcal{B}, \mu)$ is separable (has a countable dense set)

- A separable Hilbert space has a basis of orthogonal elements $\{e_n\}_1^\infty$ with all $e_n \neq 0$, $\langle e_n, e_m \rangle = 0$ when $n \neq m$ and $\langle v, e_m \rangle = 0$ for all $m \Rightarrow v = 0$.

- In this case one can $\boxed{\text{uniquely}}$ represent any $\alpha \in L^2(\mu)$ as $\alpha = \sum_{n=1}^{\infty} a_n e_n$ with $a_n = \frac{\langle \alpha, e_n \rangle}{\langle e_n, e_n \rangle}$

where this means

$$\lim_{m\to\infty} \left\| \sum_{n=1}^{m} a_n e_n \right\|_2 = 0.$$ NOTE

the convergence is in L^2 not pointwise when e_n are functions

- Let μ be Lebesque on S^1 then the exponentials $\{e^{2\pi i n\theta}\}_{n\in\mathbb{Z}}$ give an orthogonal basis for $L^2(\mu)$ and

$$\alpha = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\theta} \quad \text{with } a_n = \langle \alpha, e^{2\pi i n\theta}\rangle = \int \alpha\, e^{-2\pi i n\theta}\, d\mu$$

or $d\theta$

is the Fourier series.

- More generally, $\mathbb{T}^m = S^1 \times \dots \times S^1$ with the product measure μ_m coming from Lebesgue on S^1 and the Borels has

$$\left\{ e^{2\pi i n_1 \theta_1} e^{2\pi i n_2 \theta_2} \cdots e^{2\pi i n_m \theta_m} \right\}_{\vec{n} \in \mathbb{Z}^m}$$

as an orthogonal basis for $L^2(\mathbb{T}^m, \mathcal{B}, \mu_m)$.

Theorem: $R_\omega : S^1 \circlearrowleft$ given by $R_\omega(\theta) = \theta + \omega$ is ergodic iff $\omega \notin \mathbb{Q}$.

Proof: We use part (iv) of last time to prove both directions. To start assume $\omega \notin \mathbb{Q}$ and $\alpha \in L^2(\mu)$ has $\alpha \circ R_\omega = \alpha$.

Say the Fourier series of α is $\sum a_n e^{2\pi i n\theta}$

and so the series of $\alpha \circ R_\omega(\theta) = \alpha(\theta+\omega)$

$$= \sum a_n e^{2\pi i n(\theta+\omega)} = \sum \left(a_n e^{2\pi i n\omega}\right) e^{2\pi i n\theta}$$

Since $\alpha \circ R_\omega = \alpha$ by assumption, uniqueness of Fourier coefficients yiels $a_n = a_n e^{2\pi i n\omega}$ for all n.

Since $\omega \notin \mathbb{Q}$, $e^{2\pi i n\omega} \neq 1$ except when $n=0$.

Thus $a_n = 0$ when $n \neq 0$ so $\alpha(\theta) = a_0$ as a Fourier expansion. This implies $\alpha \in L^2(\mu)$ is the equivalence class of a constant, so it is constant a.e.

Now conversed, say $\omega = p/q \in \mathbb{Q}$, Let $\alpha(\theta) = e^{2\pi i q\theta}$

Then $\alpha \circ R_{p/q}(\theta) = e^{2\pi i q(\theta + p/q)} = e^{2\pi i q\theta} e^{2\pi i p} = e^{2\pi i q\theta} = \alpha$

is a non constant, invariant function. ∎

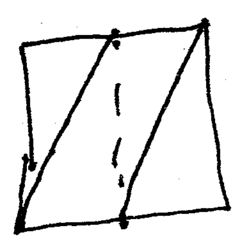
- A vector $\vec{\alpha} = (\alpha_1, \alpha_2) \in \mathbb{R}^2$ is called rationally independent if $m\alpha_1 + n\alpha_2 \in \mathbb{Z}$ $(m,n) \in \mathbb{Z}^2$ iff $m = n = 0$

- Example: $(\sqrt{2}, \frac{3}{2}\sqrt{2})$ is rationally dependent and $(\sqrt{3}, \sqrt{2})$ is rationally independent

- <u>Theorem</u> $R_{\vec{\alpha}} : \mathbb{T}^2 \to$ via $(\theta_1, \theta_2) \mapsto (\theta_1 + \alpha_1, \theta_2 + \alpha_2)$ is ergodic $\Leftrightarrow$ $\vec{\alpha}$ is rationally independent

- Proof: Two-dimensional Fourier series

- Example $\vec{\alpha} = (\sqrt{2}, 2\sqrt{2})$ has orbits on circles that wrap twice in one direction and once in the other. On ~~the~~ each circle the map induces an irrational rotation.



. For the next example we need another fact about Fourier series

. With the set up as above, if $\{e_n\}$ is an orthonormal basis $(\langle e_i, e_j \rangle = \delta_{ij})$

then if $v = \sum a_i e_i$

$$\Rightarrow \|v\|_2^2 = \sum |a_i^2|$$

. This is called Parseval's formula and sometimes the "generalized Pythogrean Theorem"

. In finite dimensions if $\vec{v} = \sum b_i q_i$ with q_i an orthonormal basis, then

$$\|\vec{v}\|_2^2 = \langle \vec{v}, \vec{v} \rangle = \langle \sum b_i q_i, \sum b_i q_i \rangle$$

$$= \sum b_i \bar{b}_i + \text{terms in } q_i q_j \text{ with } i \neq j = \sum |b_i|^2$$

- In particular, if $\{e_i\}$ is just orthogonal and $\vec{v} = \sum a_i e_i \Rightarrow \sum |a_i|^2 < \infty$

- So if $\alpha(\theta) = \sum_{n\in\mathbb{Z}} a_n e^{2\pi i n\theta} \Rightarrow \sum |a_n|^2 < \infty$.

- **Theorem**: $D: S^1 \to$ given by $D(\theta) = p\theta \mod 1$ with $|p| > 1$ is ergodic w.r.t. Lebesgue μ.

Proof: Assume $\alpha \in L^2(\mu)$ with $\alpha \circ D = \alpha$ and α non constant. We will obtain a contradiction

a.e.

If α has Fourier series

$$\alpha(\theta) = \sum a_n e^{2\pi i n\theta}$$

then $$\alpha \circ D(\theta) = \sum a_n e^{2\pi i n p\theta}$$

Since we are assuming α non constant a.e. for some $k \neq 0$, $a_k \neq 0$. By uniqueness of Fourier coefficients using

$\alpha \circ D(\theta)$ has a term $a_k e^{2\pi i k p\theta}$

and so $a_{kp} = a_k$. Continuing, $a_{kp^2} = a_{kp}$

and so $0 \neq a_k = a_{kp} = a_{kp^2} = a_{kp^3} = \cdots$

Thus $\sum |a_k|^2 = \infty$ so $\alpha \notin L^2(\mu)$ by Parsival, a contradiction. $\blacksquare$

Similar methods yield:

- **Theorem**: $f: \mathbb{T}^2 \to \mathbb{T}^2$ induced by $F\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ on $\mathbb{R}^2$

is ergodic

- Similar methods yield ergodicity ~~on of~~ for transformations on other topological groups which preserve Haar measure using Harmonic Analysis

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All the examples we have seen are also continuous functions on the spaces in question and the σ-algebra is the Borels and the measure is positive on open sets so the measure "sees" all open sets and thus the topology

Theorem Let X be a compact metric space, $f: X \to X$ is onto and continuous and μ is an invariant, ergodic Borel probability measure μ positive on open sets. Then a.e. point $x \in X$ has the property that $O^+(x,f)$ is dense in X, i.e. $\overline{O^+(x,f)} = X$

Proof Let $\{U_i\}_{i=0}^{\infty}$ be a countable base for the topology. Note that $Z \subseteq X$ is dense iff $Z \cap U_i \neq \emptyset$ for all i. Fix any i and let

$$W_i = \bigcup_{k=0}^{\infty} f^{-k}(U_i).$$

By hypothesis $\mu(U_i) > 0$ and so by ergodicity, W_i has full measure and note that $x \in W_i$ implies $f^k(x) \in U_i$ for some $k \geq 0$ i.e. $O^+(x,f) \cap U_i \neq \emptyset$

Finally, $\bigcap_{i=1}^{\infty} W_i$ is full measure and every point in it has $O^+(x, f) \cap U_i \neq \emptyset$, $\forall i$ ∎

<u>Examples</u> $R_\omega : S^1 \circlearrowleft$ with $\omega \notin \mathbb{Q}$ has every point dense by $\theta \to p\theta$ on S^1 and $f : \mathbb{T}^2 \circlearrowleft$ induced by $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ has many non-dense orbits, in fact these have the set of periodic orbits. (a single periodic orbit is not dense) is dense.

<u>Terminology</u>: A map with a dense orbit is called <u>topologically transitive</u>.

Also, a Bernoulli measure with all p_i > 0 is ergodic and positive on open sets