The MAXIMAL ergodic Theorem

- $f: (X, \mathcal{B}, \mu) \circlearrowleft$ is a mpt. It induces $U: L^p(X, \mathcal{B}, \mu) \circlearrowleft$ via $U(\alpha) = \alpha \circ f$

- By the change of variables formula:

$$\|U\|_p = \max_{\alpha \neq 0} \frac{\|U\alpha\|_p}{\|\alpha\|_p} = \max_{\alpha \neq 0} \frac{\left(\int |\alpha \circ f|^p \, d\mu\right)^{1/p}}{\left(\int |\alpha|^p \, d\mu\right)^{1/p}} = 1$$

- Also, L is a positive operator $L: L^p_{\mathbb{R}}(\mu) \circlearrowleft$ if $\alpha \geq 0 \Rightarrow L\alpha \geq 0$

2

- $L(\alpha) = \alpha \circ f$ is a positive operator
- Main used positive operators:

  $\alpha \leq \beta$ (i.e. $\alpha(x) \leq \beta(x)$ a.e.)

  $$\Rightarrow \beta - \alpha \geq 0 \Rightarrow L(\beta - \alpha) \geq 0 \Rightarrow L(\beta) - L(\alpha) \geq 0$$

  $$\Rightarrow L(\beta) \geq L(\alpha)$$

  so L is <u>order preserving</u>.

- By definition for general L

  $$\|L\| = \max_{\alpha \neq 0} \frac{\|L\alpha\|}{\|\alpha\|}$$

  it follows immediately that

  $$\|L\alpha\| \leq \|L\| \, \|\alpha\|.$$

<u>Theorem</u>: (The maximal ergodic theorem, operator version.)

Assume $U: L^1_{\mathbb{R}}(\mu) \circlearrowleft$ is a positive, linear operator with $\|U\| \leq 1$. Given $N > 0$ and $\alpha \in L^1_{\mathbb{R}}(\mu)$ Let

- $\alpha_0 = 0$
- $\alpha_n = \alpha + U\alpha + U^2\alpha + \dots + U^{n-1}\alpha \qquad n \geq 1$
- $F_N = \max\limits_{0 \leq n \leq N} \alpha_n$

$$\Rightarrow \int_{\{x : F_N(x) > 0\}} \alpha \, d\mu \geq 0$$

Proof Since $\alpha_0 = 0$, $F_N \geq 0$ and by construction $F_N \in L^1_{\mathbb{R}}(\mu)$. LF

Also by construction, $F_N \geq \alpha_n$ for all $0 \leq n \leq N$ and so since U is a positive operator, $U(F_N) \geq U(\alpha_n)$

Now $U(\alpha_n) = U\alpha + U^2\alpha + \dots + U^n\alpha$

$= \alpha_{n+1} - \alpha$

Thus $U(F_N) \geq \alpha_{n+1} - \alpha$

and so $U(F_N) + \alpha \geq \alpha_{n+1}$. Thus restricting to $A = \{x; F_N(x) > 0\}$, $U(F_N) + \alpha \geq \max_{1 \leq n \leq N} \alpha_n$

$= \max_{0 \leq n \leq N} \alpha_n$ (since we are in A)

$= F_N$

Thus $\alpha \geq F_N - UF_N$ on the set A.

Thus $\int_A \alpha \, d\mu \geq \int_A F_\nu \, d\mu - \int_A u F_\nu \, d\mu$

$= \int_X F_\nu \, d\mu - \int_A u F_\nu \, d\mu$ since $F_\nu = 0$ on $X - A$ (recall $F_\nu \geq 0$)

$\geq \int_X F_\nu \, d\mu - \int_X u F_\nu \, d\mu$

$F_\nu \geq 0$ so $u F_\nu \geq 0$ thus $\int_X u F_\nu \, d\mu \geq \int_A u F_\nu \, d\mu$

≥ 0

Since $\|u\| \leq 1$ so $\|u F_\nu\| \leq \|F_\nu\|$, $\|u F_\nu\| = \int u F_\nu \, d\mu$, $\|F_\nu\| = \int F_\nu \, d\mu$, $F_\nu \geq 0$, $u F_\nu \geq 0$

as required ∎

<u>CORR</u> (the Concrete MAXIMAL Ergodic Theorem)

$f : X \circlearrowleft$ is mpt $\alpha \in L^1_{\mathbb{R}}(\mu)$ and

$$B_a = \left\{ x \in X : \sup_{n \geq 1} \frac{1}{n} \sum_{i=0}^{n-1} \alpha(f^i(x)) > a \right\}$$

then if $f^{-1}(A) = A$

$$\int_{B_a \cap A} \alpha \, d\mu \geq a \, \mu(B_a \cap A)$$

<u>Proof</u>. In the theorem the U here is $U(\alpha) = \alpha \circ f$

Let $\beta = \alpha - a$, construct F_N from β

$$\beta_n = \beta + U\beta + \cdots + U^{n-1}\beta$$

$$F_N = \max_{0 \leq n \leq N} \beta_n$$

Now consider the sum in B_a

$$\frac{1}{n}\sum_{i=0}^{n-1} \alpha(f^i(x)) > a$$

or $\frac{1}{n}\sum U^i \alpha > a$

or $\frac{1}{n}\sum U^i(\beta + a) > a$

or $\left(\frac{1}{n}\sum U^i(\beta)\right) + a > a$ since $U^i a = a$.

or $\underbrace{\sum_{i=0}^{n-1} U^i(\beta)}_{\beta_n} > 0$ and $F_N = \max \beta_n$

Thus $B_a = \bigcup_{N=0}^{\infty} \{x : F_N(x) > 0\}$

But by the theorem, $\int_{\{F_N > 0\}} \beta \, d\mu > 0$

Thus $\int_{B_a} \beta \, d\mu > 0$ or $\int_{B_a} (\alpha - a) \, d\mu > 0$ or $\int_{B_a} \alpha \, d\mu > \int_{B_a} a \, d\mu = a(\mu(B_a))$

That proves the theorem with $A = X$.
To get the actual statement, since $f^{-1}(A) = A$
we can restrict to $f|_A$ and applying the
result there finishes it. $\boxtimes$

We begin the proof of the pointwise ergodic theorem

Theorem: $f : (X, \mathcal{B}, \mu)$ is mpt of probability space
$\alpha \in L^1(\mu) \Rightarrow \exists\, \alpha^* \in L^1(\mu)$ such that

(1) $\lim_{n \to \infty} \frac{1}{n} \sum_{i=0}^{n-1} \alpha(f^i(x)) \to \alpha^*(x)$ a.e.

(2): $\alpha^* \circ f = \alpha^*$ a.e.

(3) $\int \alpha^* \, d\mu = \int \alpha \, d\mu$.

Proof: To start, let $S_n(x) = \frac{1}{n}\sum_{i=0}^{n-1} \alpha \circ f^i(x)$

and $$\alpha^*(x) = \limsup_{n\to\infty} S_n(x)$$

$$\alpha_*(x) = \liminf_{n\to\infty} S_n(x)$$

We need to show

(a) $\alpha^* = \alpha_*$ a.e. and is in $L^1(\mu)$

(b) $\alpha^* \circ f = \alpha^*$ a.e.

(c) $\int \alpha^* \, d\mu = \int \alpha \, d\mu$.

We first show (b) for both α_* and α^*.

We claim

$$\left(\frac{n+1}{n}\right) S_{n+1}(x) - S_n(f(x)) = \frac{\alpha(x)}{n}$$

Thus $\limsup\limits_{n\to\infty} S_{n+1}(x) - \limsup\limits_{n\to\infty} S_n(f(x)) = 0$

or $\alpha^*(x) = \alpha^*(f(x))$ similarly for α_*.

The claim is algebra

$$\left(\frac{n+1}{n}\right) S_{n+1}(x) - S_n(f(x))$$

$$= \frac{n+1}{n}\left(\frac{1}{n+1}\sum_{i=0}^{n} \alpha \circ f^i(x)\right) - \frac{1}{n}\sum_{i=0}^{n-1} \alpha\, f^{i+1}(x)$$

$$= \frac{1}{n}\sum_{i=0}^{n} \alpha \circ f^i(x) - \frac{1}{n}\sum_{i=0}^{n-1} \alpha\, f^{i+1}(x)$$

$$= \frac{1}{n}\sum_{i=0}^{n} \alpha \circ f^i(x) - \frac{1}{n}\sum_{i=1}^{n} \alpha\, f^{i}(x) \quad \text{(shifting indices)}$$

$$= \frac{\alpha(x)}{n}.$$

Proof continued next time.