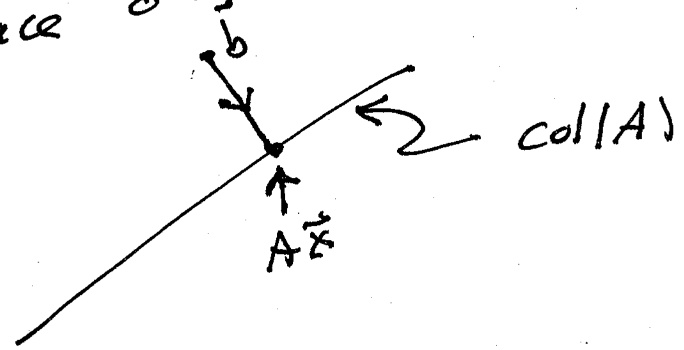
(1) (3pts) If A is an $n \times n$ symmetric matrix, $A^T = A$, then

(a) All the eigenvalues of A are real

(b) There exists an orthogonal matrix Q so that $Q^TAQ = \text{diag}(\lambda_1, \ldots, \lambda_n)$ with λ_i the eigenvalues of A. Equivalently, there exists an orthonormal set $\{\vec{q}_1, \ldots, \vec{q}_n\}$ of eigen vectors for A

(2) (7pts) we seek $\vec{x}$ so that $A\vec{x}$ is a close as possible to $\vec{b}$ in the two norm. Geometrically, this means that the vector from $\vec{b}$ to Ax is orthogonal to the column space of A



or $(A\vec{x} - \vec{b}) \perp \text{col}(A)$. In terms of matrices

$$A^T(A\vec{x} - \vec{b}) = 0$$ since $A\vec{x} - \vec{b} \perp$ each col of A and thus each row of A^T

multiplying out we get $A^TA\vec{x} - A^T\vec{b} = 0$ or

$$A^TA\vec{x} = A^T\vec{b}$$

(3) (5 pts) The thin SVD of A is $A = U\Sigma V^T$ with U $m \times n$ with orthogonal columns (so $U^TU = I$) and $\Sigma = \mathrm{diag}(\sigma_1, \ldots, \sigma_n)$ with all $\sigma_i > 0$ since A is full rank and V is $n \times n$ and orthogonal.

Thus $(A^TA) = V\Sigma^T U^T U \Sigma V^T$

$$= V\Sigma^T\Sigma V^T = V\,\mathrm{diag}(\sigma_1^2, \ldots, \sigma_n^2)\,V^T$$

and so $(A^TA)^{-1} = V\,\mathrm{diag}(\sigma_1^{-2}, \ldots, \sigma_n^{-2})\,V^T$

(4) (3 pts) A has r nonzero singular values

(5) (3) A^TA has singular values $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$

(6) (7 pts) $A = \hat{Q}\hat{R}$, $A^+ = (A^TA)^{-1}A^T$

$$= (\hat{R}^T\hat{Q}^T\hat{Q}\hat{R})^{-1}\hat{R}^T\hat{Q}^T = (\hat{R}^T\hat{R})^{-1}\hat{R}^T\hat{Q}^T$$

$$= \hat{R}^{-1}\hat{R}^{-T}\hat{R}^T\hat{Q}^T = \hat{R}^{-1}\hat{Q}^T$$

where we used the facts that $\hat{Q}$ has orthonormal columns and so $\hat{Q}^TQ = I$ and since A is full rank, $\hat{R}$ and $\hat{R}^T$ are invertible.

S1

7) $B_1 = 10\sqrt{10} \begin{bmatrix} 1/\sqrt{10} \\ 2/\sqrt{10} \\ 1/\sqrt{10} \\ 2/\sqrt{10} \end{bmatrix} \begin{bmatrix} \frac{3}{5} & -\frac{4}{5} & 0 \end{bmatrix}$

$= 10\sqrt{10} \begin{bmatrix} \frac{3}{5\sqrt{10}} & -\frac{4}{5\sqrt{10}} & 0 \\ \frac{6}{5\sqrt{10}} & -\frac{8}{5\sqrt{10}} & 0 \\ \frac{3}{5\sqrt{10}} & -\frac{4}{5\sqrt{10}} & 0 \\ \frac{6}{5\sqrt{10}} & -\frac{8}{5\sqrt{10}} & 0 \end{bmatrix}$

$= \begin{bmatrix} 6 & -8 & 0 \\ 12 & -16 & 0 \\ 6 & -8 & 0 \\ 12 & -16 & 0 \end{bmatrix}$

$$B_2' = 5\sqrt{2}\begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \\ 0 \end{bmatrix}\begin{bmatrix} +4/5 & 3/5 & 0 \end{bmatrix}$$

$$= 5\sqrt{2}\begin{bmatrix} \frac{+4}{5\sqrt{2}} & \frac{3}{5\sqrt{2}} & 0 \\ 0 & 0 & 0 \\ -\frac{4}{5\sqrt{2}} & \frac{-3}{5\sqrt{2}} & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} +4 & 3 & 0 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$B_2 = B_1 + B_2' = \begin{bmatrix} 6 & -8 & 0 \\ 12 & -16 & 0 \\ 6 & -8 & 0 \\ 12 & -16 & 0 \end{bmatrix} + \begin{bmatrix} +4 & 3 & 0 \\ 0 & 0 & 0 \\ -4 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 10 & -5 & 0 \\ 12 & -16 & 0 \\ 2 & -11 & 0 \\ 12 & -16 & 0 \end{bmatrix}$$

$$= \cancel{\begin{bmatrix} 2 & 11 & 0 \\ 12 & 16 & 0 \\ 10 & 6 & 0 \\ 12 & 16 & 0 \end{bmatrix}} \qquad \cancel{\begin{bmatrix} 10 & 11 & 0 \\ 12 & 16 & 0 \\ 2 & 5 & 0 \end{bmatrix}}$$

$\lambda_L = \sigma_L^2$ so sing val are 1000, 50, 4

8) $A^T A = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}$

$= \begin{bmatrix} 5 & -3 \\ -3 & 5 \end{bmatrix}$

$p(\lambda) = \begin{vmatrix} 5-\lambda & -3 \\ -3 & 5-\lambda \end{vmatrix} = (5-\lambda)^2 - 9$

$= \lambda^2 - 10\lambda + 16 = (\lambda - 8)(\lambda - 2), \quad \lambda = 8, 2$

So $\sigma_1 = \sqrt{8} = 2\sqrt{2}$

$\sigma_2 = \sqrt{2}$

E vect of $A^T A$

$\lambda = 8$, $\begin{bmatrix} -3 & -3 \\ -3 & -3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0} \quad \vec{v}_1' = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\vec{v}_1 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$

$\lambda = 2 \quad \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \vec{0}, \quad \vec{v}_2' = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\vec{v}_2 = \begin{bmatrix} 1/\sqrt{2} \\ +1/\sqrt{2} \end{bmatrix}$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{\begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ -1/\sqrt{2}\end{pmatrix}}{2\sqrt{2}} = \frac{\begin{bmatrix}0\\ 2\sqrt{2}\end{bmatrix}}{2\sqrt{2}} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{\begin{pmatrix}1 & 1\\ 2 & -2\end{pmatrix}\begin{pmatrix}1/\sqrt{2}\\ 1/\sqrt{2}\end{pmatrix}}{\sqrt{2}} = \frac{\begin{bmatrix}2/\sqrt{2}\\ 0\end{bmatrix}}{\sqrt{2}} = \begin{bmatrix}1\\ 0\end{bmatrix}$$

so $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\sqrt{8} & 0\\ 0 & \sqrt{2}\end{bmatrix}\underbrace{\begin{bmatrix}1/\sqrt{2} & -1/\sqrt{2}\\ +1/\sqrt{2} & 1/\sqrt{2}\end{bmatrix}}_{V^T}$ (9)

(for b) $\kappa(A) = \frac{\sqrt{8}}{\sqrt{2}} = 2.$

(9) $A = \begin{bmatrix}2 & 2\\ 2 & 4\\ 2 & 2\\ 2 & 4\end{bmatrix}$

$$q_1 = \frac{\begin{bmatrix}2\\ 2\\ 2\\ 2\end{bmatrix}}{\sqrt{16}} = \begin{bmatrix}\frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\\ \frac{1}{2}\end{bmatrix}$$

$$\vec{v_2} = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 4 \end{bmatrix} - \left[\begin{pmatrix} 2 \\ 4 \\ 2 \\ 4 \end{pmatrix} \cdot \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \right] \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{matrix} 2 \\ 4 \\ 2 \\ 4 \end{matrix} - 8 \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\vec{q_2} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \qquad \hat{Q} = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}$$

$$A = \hat{Q}\hat{R} \Rightarrow \hat{R} = \hat{Q}^T A =$$

$$\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 2 & 4 \\ 2 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 0 & 2 \end{bmatrix}$$

(b) From 1 we know $A^+ = \hat{R}^{-1} \hat{Q}^T$

$$= \frac{1}{8} \begin{bmatrix} 2 & -6 \\ 0 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$$

$$= \frac{1}{8}\begin{bmatrix} 4 & -2 & 4 & -2 \\ -2 & 2 & -2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/4 & 1/2 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \end{bmatrix}$$

~~(b)~~ So $\vec{x} = A^+\vec{b} = \begin{bmatrix} 1/2 & -1/4 & 1/2 & -1/4 \\ -1/4 & 1/4 & -1/4 & 1/4 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ 4 \\ -2 \end{bmatrix}$

$$= \begin{bmatrix} 1 - 1 + 2 + 1/2 \\ \frac{-1}{2} + 1 - 1 - 1/2 \end{bmatrix} = \begin{bmatrix} 5/2 \\ -1 \end{bmatrix}$$

(c) $P = \hat{Q}\hat{Q}^T = \begin{bmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & 1/2 & -1/2 & 1/2 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$