1) $\|\vec{v} - \vec{v}^{(k)}\|_2^2 = \left\| \sum_{i=1}^{N} \alpha_i \vec{q}_i - \sum_{i=1}^{k} \alpha_i \vec{q}_i \right\|_2^2$

$= \left\| \sum_{i=k+1}^{N} \alpha_i \vec{q}_i \right\|_2^2 = \sum_{i=k+1}^{N} |\alpha_i|^2$ using Pythag.

(2) $w_1 = 1, \ \vec{q}_1 = \frac{1}{\|1\|} = \frac{1}{\sqrt{\int_{-1}^{1} 1^2 dt}} = \frac{1}{\sqrt{2}}$

$w_2 = t, \ \vec{v}_2 = \vec{w}_2 - \langle \vec{w}_2, \vec{q}_1 \rangle \vec{q}_1$

$= t - \left(\int_{-1}^{1} t \cdot \frac{1}{\sqrt{2}} dt \right) \frac{1}{\sqrt{2}}$

$= t - 0 = t$

$\vec{q}_2 = \frac{t}{\|t\|} = \frac{t}{\sqrt{\int_{-1}^{1} t^2 dt}} = \frac{t}{\sqrt{\frac{2}{3}}} = \sqrt{\frac{3}{2}}\, t$

$w_3 = t^2, \ \vec{v}_3 = \vec{w}_3 - \langle \vec{w}_3, \vec{q}_1 \rangle q_1 - \langle \vec{w}_3, \vec{q}_2 \rangle \vec{q}_2$

$= t^2 - \left(\int_{-1}^{1} t^2 \cdot \frac{1}{\sqrt{2}} \right) \frac{1}{\sqrt{2}} - \left(\int_{-1}^{1} t^2 \cdot \sqrt{\frac{3}{2}}\, t \right) \sqrt{\frac{3}{2}}\, t$

$= t^2 - \frac{2}{3} \cdot \frac{1}{2} - 0 = t^2 - 1/3$

$\vec{q}_3 = \frac{t^2 - 1/3}{\sqrt{\int_{-1}^{1} (t^2 - 1/3)^2 dt}} = \frac{t^2 - 1/3}{\left(\int_{-1}^{1} t^4 - \frac{2}{3} t^2 + 1/9 \right)^{1/2}} = \frac{t^2 - 1/3}{\sqrt{8/45}}$

$= \frac{3}{2} \sqrt{\frac{5}{2}} \, (t^2 - 1/3)$

(3) (a) $\omega_N^N = \left(e^{2\pi i/N}\right)^N = e^{2\pi i N/N} = e^{2\pi i} = 1$

$\omega_N = \cos\frac{2\pi}{N} + i\sin\frac{2\pi}{N}$, $|\omega_N|^2 = \cos^2\frac{2\pi}{N} + \sin^2\frac{2\pi}{N} = 1$

$(\omega^k)^N = (\omega^N)^k = 1^k = 1$

$\bar{\omega}^k = \overline{(\omega^k)} = \overline{e^{2\pi i k/N}} = \overline{\cos\frac{2\pi k}{N} + i\sin\frac{2\pi k}{N}}$

$= \cos\frac{2\pi k}{N} - i\sin\frac{2\pi k}{N} = \cos\left(-\frac{2\pi k}{N}\right) + i\sin\left(-\frac{2\pi k}{N}\right)$

$= e^{-2\pi i k/N} = \omega^{-k}.$

(b) multiply out and cancel.

first orthogonality (using (a))

$\langle \vec{z}_j, \vec{z}_k \rangle = \langle \frac{1}{\sqrt{N}}[1, \bar{\omega}_N^{j}, \ldots, \omega_N^{-(N-1)j}]^T,$

$\frac{1}{\sqrt{N}}[1, \omega_N^k, \ldots, \omega_N^{+(N-1)k}]$

$= \frac{1}{N}\left(1 + \omega_N^{(k-j)} + \omega_N^{2(k-j)} + \ldots + \omega_N^{N-1(k-j)}\right)$

$\underset{(b)}{=} \frac{1}{N}\,\frac{1-(\omega_N^{k-j})^N}{1-\omega_N^{k-j}} = 0$ since $k \neq j$

Now normality

$$\|\vec{z}_j\|^2 = \left\langle \frac{1}{\sqrt{N}}\left[1, \bar{\omega}_N^{j}, \dots, \bar{\omega}_N^{(N-1)j}\right]^T, \frac{1}{\sqrt{N}}\left[1, \omega_N^{j}, \dots, \omega_N^{(N-1)j}\right] \right\rangle$$

$$= \frac{1}{N}\left[1 + \omega_N^{j-j} + \dots + \omega_N^{(N-1)j-(N-1)j}\right] \quad \text{(using 9)}$$

$$= \frac{1}{N}\left[1 + 1 + \dots + 1\right] = \frac{N}{N} = 1$$

Now we showed orthogonal sets are linearly independent and there are N elements in the N-dimensional space $\mathbb{C}^N$, so the form a basis.

$$\cancel{(4)\ n=0,\ \left\langle f, \frac{1}{\sqrt{2\pi}} \right\rangle = \int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{2\pi}}\, dt = \frac{2\pi}{4}\frac{1}{\sqrt{2\pi}} = \frac{\pi}{2}\frac{1}{\sqrt{2\pi}}}$$

$$\cancel{n \neq 0,\ \left\langle f, \frac{e^{int}}{\sqrt{2\pi}} \right\rangle}$$

11d

(4) $n=0$, $\alpha_0 = \langle \frac{1}{\sqrt{2\pi}}, f \rangle = \int_{-\pi/4}^{\pi/4} \frac{1}{\sqrt{2\pi}}\,dt = \frac{2\pi}{4}\frac{1}{\sqrt{2\pi}} = \frac{\pi}{2}\frac{1}{\sqrt{2\pi}}$

$n \neq 0$, $\alpha_n = \langle \frac{e^{int}}{\sqrt{2\pi}}, f \rangle = \frac{1}{\sqrt{2\pi}} \int_{-\pi/4}^{\pi/4} e^{-int}\,dt$

$$= \frac{1}{in\sqrt{2\pi}}\left(e^{-in\pi/4} - e^{in\pi/4}\right)$$

$$= \frac{1}{in\sqrt{2\pi}} 2i\sin(n\pi/4) = \frac{2\sin(n\pi/4)}{n\sqrt{2\pi}}$$

so

$$f(t) \sim \left(\frac{\pi}{2}\frac{1}{\sqrt{2\pi}}\right)\cdot\frac{1}{\sqrt{2\pi}} + \left(\sum_{n\neq 0}\left(\frac{2\sin n\pi/4}{n\sqrt{2\pi}}\right)\cdot\frac{e^{int}}{\sqrt{2\pi}}\right)$$

$$= \frac{1}{4} + \sum_{n\neq 0} \frac{\sin(n\pi/4)}{n\pi} e^{int}$$

which can be simplified further, but wasn't asked for.