HW7.1

(a) $$\hat{f}(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\, e^{-ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-ist}\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \left. \frac{e^{-ist}}{-is} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{-is} \left[e^{-\pi i s} - e^{\pi i s} \right]$$

$$= \frac{1}{\sqrt{2\pi}(-is)} \left[\cos \pi s - i \sin \pi s - [\cos \pi s + i \sin \pi s] \right]$$

$$= \frac{1}{\sqrt{2\pi}(-is)} \left[-2 i \sin \pi s \right] = \boxed{\frac{2}{\sqrt{2\pi}} \frac{\sin \pi s}{s}}$$

(b) $$(\sqrt{2\pi})\hat{f}(s) = \int_{-1}^{1} t\, e^{-ist}\, dt$$

$$\boxed{\begin{array}{l} u = t,\ dv = e^{-ist} dt \\ du = dt\quad v = \dfrac{e^{-ist}}{-is} \end{array}}$$

$$= \left. \frac{t e^{-ist}}{-is} \right|_{-1}^{1} + \frac{1}{is} \int_{-1}^{1} e^{-ist}\, dt$$

$$= -\frac{1}{is}\left[e^{-is}+e^{is}\right] + \frac{1}{is}\,\frac{e^{-is}}{-is}\Bigg|_{-1}^{1}$$

$$= -\frac{1}{is}\,2\cos s + \frac{1}{s^2}\left[e^{-is}-e^{is}\right]$$

$$= +\frac{2i}{s}\,\frac{\cos s}{s} - 2i\,\frac{\sin s}{s^2}$$

$$= 2i\left[\frac{s\cos s - \sin s}{s^2}\right] \Rightarrow \hat{f}(s) = \frac{1}{\sqrt{2\pi}}\,2i\left[\frac{\cos s}{s} - \frac{\sin s}{s^2}\right]$$

(c) $$e^{-|t|} \begin{cases} = e^{-t} & t \geq 0 \\ = e^{t} & t \leq 0 \end{cases}$$

$$\text{So } \sqrt{2\pi}\,\hat{f}(s) = \int_{-\infty}^{0} e^{t} e^{-ist}\,dt + \int_{0}^{\infty} e^{-t} e^{-ist}\,dt$$

$$= \lim_{T\to-\infty}\int_{T}^{0} e^{(1-is)t}\,dt + \lim_{T\to\infty}\int_{0}^{T} e^{-(1+is)t}\,dt$$

$$= \lim_{T \to -\infty} \left. \frac{e^{(1-is)t}}{(1-is)} \right|_T^0 + \lim_{T \to \infty} \left. -\frac{e^{-(1+is)t}}{(1+is)} \right|_0^T$$

$$= \lim_{T \to -\infty} \frac{-e^{(1-is)T} + 1}{1-is} + \lim_{T \to \infty} -\left(\frac{e^{(1+is)T} - 1}{1+is}\right)$$

$$= \frac{1}{1-is} + \frac{+1}{1+is} = \frac{(1+is) + (1-is)}{(1-is)(1+is)}$$

$$= \frac{2}{1+s^2}$$

So $\hat{f}(s) = \dfrac{2}{\sqrt{2\pi}} \dfrac{1}{1+s^2}$