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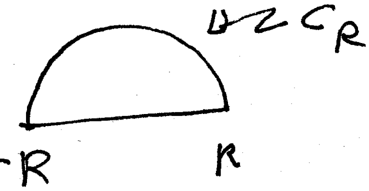
(1) $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\, PV \int_{-\infty}^{\infty} \frac{dx}{x^2+1}$ since $\frac{1}{x^2+1}$ is even

(2) let Γ_R be the contour



so

$$\oint_{\Gamma_R} \frac{dz}{z^2+1} = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1}$$

(3) Since z^2+1 has roots $\pm i$ and only i is inside the contour

$$\oint_{\Gamma_R} \frac{dz}{z^2+1} = 2\pi i \operatorname*{Res}_{z=i} \frac{1}{z^2+1} = 2\pi i \frac{p(i)}{q'(i)} = \frac{2\pi i}{2i} = \pi$$

writing $\frac{1}{z^2+1} = \frac{p(z)}{q(z)}$

(4) Since by reverse triangle inequality

$$|z^2+1| \geq ||z|^2-1| = R^2-1 \text{ on } C_R$$

using the integral triangle inequality

$$0 \leq \left| \lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2+1} \right| \leq \lim_{R\to\infty} (2\pi R)\frac{1}{R^2-1} = 0$$

so $\displaystyle\lim_{R\to\infty} \int_{C_R} \frac{dz}{z^2+1} = 0$

(5) Taking the limit in (2) $\pi = PV \displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1}$

so by (1) $\displaystyle\int_0^{\infty} \frac{dx}{x^2+1} = \frac{1}{2}\pi$

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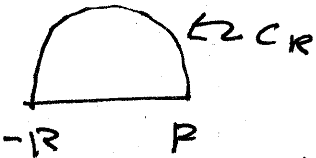
(1) $\dfrac{\cos x}{(x^2+a^2)(x^2+b^2)}$ is even so $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} = PV\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+a^2)(x^2+b^2)}$

(2) Let Γ_R be the contour



so $\displaystyle\oint_{\Gamma_R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)} = \int_{-R}^{R} \frac{e^{ix}\,dx}{(x^2+a^2)(x^2+b^2)} + \int_{C_R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}$

Taking the real part

$$\mathrm{Re}\int_{\Gamma_R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)} = \int_{-R}^{R} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}\,dx + \mathrm{Re}\int_{C_R}\frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)}$$

(3) The roots of the denom are $\pm ai$ and $\pm bi$. When $R > a, b$ then only ai and bi are inside Γ_R

Letting $p(z) = e^{iz}/(z^2+b^2)$, $q(z) = z^2+a^2$ then $\displaystyle\operatorname*{Res}_{z=ai} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{-a}/b^2-a^2}{2ai}$

Letting $p(z) = e^{iz}/z^2+a^2$, $q(z) = z^2+a^2$ then $\displaystyle\operatorname*{Res}_{z=bi} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} = \frac{e^{-b}/a^2-b^2}{2bi}$

so $\displaystyle\oint_{\Gamma_R} \frac{e^{iz}\,dz}{(z^2+a^2)(z^2+b^2)} = 2\pi i\left[\frac{e^{-a}}{(b^2-a^2)2ai} + \frac{e^{-b}}{(a^2-b^2)2bi}\right]$

$$= \frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right]$$

(4) Now $\left| \operatorname{Re} \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} \right| \le \left| \int_{C_R} \frac{e^{iz}}{(z^2+a^2)(z^2+b^2)} \right|$

Let $f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ which is analytic when $R > \max\{a,b\}$

and by the reverse triangle inequality $|z^2+a^2| \ge ||z^2| - |a^2||$

$= R^2 - a^2$ on C_R and similarly, $|z^2+b^2| \ge R^2 - b^2$ on C_R

so $|f(z)| \le \frac{1}{(R^2-a^2)(R^2-b^2)} \to 0$ as $R \to \infty$

so by Jordan's lemma $\lim_{R\to\infty} \int_{C_R} \frac{e^{iz}\, dz}{(z^2+a^2)(z^2+b^2)} = 0$

(5) Taking the limit in (2)

$$\frac{\pi}{a^2-b^2}\left[\frac{e^{-b}}{b} - \frac{e^{-a}}{a}\right] = PV \int_{-R}^{R} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}$$

$$= \int_{-\infty}^{\infty} \frac{\cos x\, dx}{(x^2+a^2)(x^2+b^2)}$$