Solving ~~A~~ Matrix DE using evals + vect continued

$\boxed{\text{Solve}}$ $\dfrac{d\vec{x}}{dt} = A\vec{x}$ with $\vec{x}(0)$ given

A has evect $\{\vec{v}_1, \dots, \vec{v}_n\}$ which form a basis for $\mathbb{R}^n$ with eval $(\lambda_1, \dots, \lambda_n)$

$$A = X \Lambda X^{-1}$$

$$\Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

$$X = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = A\vec{x} = X \Lambda X^{-1}\vec{x}$$

$$\frac{d X^{-1}\vec{x}}{dt} = \Lambda X^{-1}\vec{x}, \quad \underbrace{\vec{y} = X^{-1}x}_{\text{new coordinates}}$$

$$\frac{d\vec{y}}{dt} = \Lambda \vec{y}$$ ← decoupled

$$\Rightarrow \quad y(t) = e^{\Lambda t}\vec{y}(0) \quad \text{where}$$

$$e^{\Lambda t} = \mathrm{diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_n t}\right)$$

BACK TO x coord.

so $X^{-1} x(t) = e^{\Lambda t} X^{-1} \vec{x}_0$ | $\vec{x}_0 = \vec{x}(0)$

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$$x(t) = X e^{\Lambda t} X^{-1} \vec{x}_0 \quad (*)$$

Aside: what does $X^{-1}\vec{x}_0$ mean

If $X^{-1}\vec{x}_0 = \vec{\alpha}$

then $\vec{x}_0 = X\vec{\alpha} = \left[\vec{v}_1 \dots \vec{v}_n\right]\begin{bmatrix}\alpha_1 \\ \vdots \\ \alpha_n\end{bmatrix}$

$$= \sum_{i=1}^{n} \alpha_i \vec{v}_i \quad \text{so}$$

α_i are the coordinates of $\vec{x}_0$ in the basis given by $\{\vec{v}_1, \dots, \vec{v}_n\}$

<u>Summary</u>: If $\mathbb{X} = [\vec{v}_1 \cdots \vec{v}_n]$, invertible,

and $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_n\} \Rightarrow$ for any $\vec{x}$

$$\mathbb{X}^{-1}\vec{x} = [\vec{x}]_{\mathcal{B}}$$

BACK TO DE

$$x(t) = \mathbb{X} e^{\Lambda t} \mathbb{X}^{-1} x_0$$
$$= \mathbb{X} e^{\Lambda t} \vec{\alpha} \text{ with}$$

$$\vec{\alpha} = [\vec{x}_0]_{\mathcal{B}}$$

$$\vec{x}(t) = \underline{X}\, e^{\Lambda t}\, \vec{\alpha} \leftarrow \underline{\text{matrix form}}$$

$$= \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_n \end{bmatrix} \begin{bmatrix} e^{\lambda_1 t} & & 0 \\ & \ddots & \\ 0 & & e^{\lambda_n t} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda_1 t}\vec{v}_1 & \cdots & e^{\lambda_n t}\vec{v}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{bmatrix}$$

$$= \sum_{i=1}^{n} \alpha_i e^{\lambda_i t} \vec{v}_i \leftarrow \underline{\text{vector form}}$$

where $\vec{\alpha} = \left[\vec{x}(0)\right]_{\mathcal{B}} = \underline{X}^{-1}\vec{x}(0)$

Solve $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 8 & 3 \\ 2 & 7 \end{bmatrix} \vec{x}$ with $\vec{x}(0) = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$

In both matrix and vector Form.

Previously we computed.

$$\Phi = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \quad \Lambda = \begin{bmatrix} 10 & 0 \\ 0 & 5 \end{bmatrix}$$

By (*)

$$x(t) = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} e^{10t} & 0 \\ 0 & e^{5t} \end{bmatrix} \left(\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \end{bmatrix} \right)$$

How do we compute

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = \vec{\alpha}$$

Solve $\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \end{bmatrix}$ by LU or elimination ...

2x2 matrix formula:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 7 \\ 3 \end{bmatrix} = -\frac{1}{5} \begin{bmatrix} -1 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 7 \\ 3 \end{bmatrix}$$

$$= -\frac{1}{5} \begin{bmatrix} -10 \\ -5 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

matrix form $\vec{x}(t) = \begin{bmatrix} 3 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} e^{10t} & 0 \\ 0 & e^{5t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

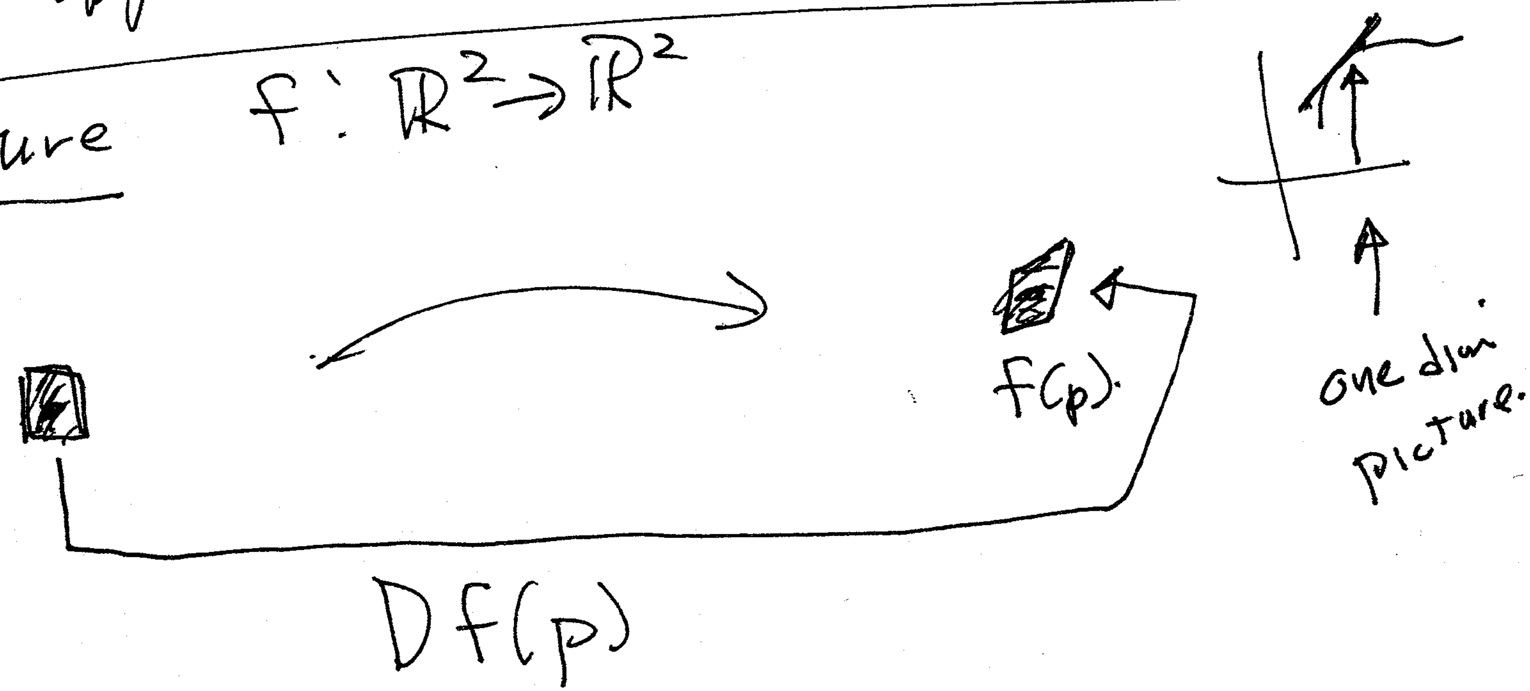
vector form $\vec{x}(t) = 2e^{10t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} + e^{5t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

MAke sure eigenvector
and value match

One OF important roles of Lin. Alg.
is in multivariable calculus $\longleftrightarrow$
the derivative is a matrix

Big idea: The derivative is the best, local, linear approx to a differentiable function

Picture $f : \mathbb{R}^2 \to \mathbb{R}^2$



$f: \mathbb{R}^n \to \mathbb{R}^m$

$f = (f_1, \ldots, f_m)$

$f_i(x_1, \ldots, x_n) \quad i = 1, \ldots, m$

$$Df(p) = \frac{\partial f_i}{\partial x_j}(p) = \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(p) & \cdots & \frac{\partial f_1}{\partial x_n}(p) \\ \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(p) & \cdots & \frac{\partial f_m}{\partial n}(p) \end{bmatrix}$$

$\boxed{m \times n \text{ matrix}}$ also written

∇f, Jf or $\frac{\partial \vec{f}}{\partial \vec{x}}$

Called Derivative matrix, Jacobian matrix. (The "Jacobian" is usuall $\det(Df(p))$)

example

$$f(x_1, x_2) = (\sin x_1 + x_2^2, \cos x_1 - 2x_2)$$

$$Df(x_1, x_2) = \begin{bmatrix} \cos x_1 & 2x_2 \\ -\sin x_1 & -2 \end{bmatrix}$$

$$Df\left(\frac{\pi}{4}, 1\right) = \begin{bmatrix} \sqrt{2}/2 & 2 \\ -\frac{\sqrt{2}}{2} & -2 \end{bmatrix}$$

Formalizes the big idea = Taylor's Theorem.

$$\underbrace{f(\vec{x}+\overrightarrow{\Delta x})}_{\text{local coord}} = f(\vec{x}) + \underbrace{Df(\vec{x})\,\overrightarrow{\Delta x}}_{\text{Loc. Lin. approx}} + \underbrace{h.o.t.}_{\text{higher order terms.}}$$

An affine map is of the form

$$\boxed{h(\vec{x}) = A\vec{x} + \vec{b}}$$

Find the best affine approximation to the example $f(x)$ at the point $\left(\frac{\pi}{4}, 1\right)$.

$$h(\vec{y}) = f\left(\frac{\pi}{4}, 1\right) + Df\left(\frac{\pi}{4}, 1\right)\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= \begin{bmatrix} \frac{\sqrt{2}}{2} + 1 \\ \frac{\sqrt{2}}{2} - 2 \end{bmatrix} + \begin{bmatrix} \sqrt{2}/2 & 2 \\ -\frac{\sqrt{2}}{2} & -2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

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special case $\Phi : \mathbb{R}^m \to \mathbb{R}$, cost function, optimization, objective function

Taylor's Theorem for $\Phi : \mathbb{R}^m \to \mathbb{R}$

$$\Phi(\vec{x}+\Delta\vec{x}) = \Phi(\vec{x}) + \nabla\Phi(\vec{x})\,\Delta\vec{x} + \Delta\vec{x}^T H\Phi(x)\,\Delta\vec{x} + \text{h.o.t.}$$

where $\nabla\Phi = \left(\frac{\partial\Phi}{\partial x_1}, \ldots, \frac{\partial\Phi}{\partial x_n}\right)$

grad or nabla

$$H\Phi = \left(\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}\right) = \text{Hessian matrix}$$

If Φ is C^2 (2-continuous derivatives) then $H\Phi$ is symmetric since $\frac{\partial^2\Phi}{\partial x_i\,\partial x_j} = \frac{\partial^2\Phi}{\partial x_j\,\partial x_i}$