Symmetric Matrices and the Spectral Theorem

- $H\Phi(x_0)$ is symmetric
- another example

Covariance matrix (loosely)

$$S = A^T A, \quad S^T = (A^T A)^T = A^T A^{TT} = A^T A = S$$

$$A = \begin{bmatrix} \vec{c}_1 \cdots \vec{c}_n \\ \\ \end{bmatrix}, \quad A^T A = \begin{bmatrix} \vec{c}_1^T \\ \vdots \\ \vec{c}_n \end{bmatrix} \begin{bmatrix} \vec{c}_1 \cdots \vec{c}_n \\ \\ \end{bmatrix}$$

$$= i \rightarrow \begin{bmatrix} \vec{c}_i^T \vec{c}_j \\ \\ \end{bmatrix}$$

$(A^TA)_{ij} = \vec{C}_i \cdot \vec{C}_j$

AA^T is also symmetric

A is $m \times n \Rightarrow$

A^TA is $(n \times m) \cdot (m \times n) = n \times n$

AA^T is $(m \times n)(n \times m) = m \times m$

square, but different says when

A is not square

Square symmetric matrices are also called self-adjoint

Here's the reason

$$x \circ (S\vec{y})$$

$$\parallel$$

$$x^T S \vec{y}$$

$$\parallel$$

$$(S^T x)^T \vec{y}$$

$$\parallel$$

$$(S^T x) \circ \vec{y} = Sx \circ y$$

dot

So if S is symmetric

$$(S\vec{x}) \cdot \vec{y} = \vec{x} \cdot S\vec{y}$$

$$\langle Sx, y \rangle = \langle x, Sy \rangle$$

Spectral Theorem / orthogonally diagonalizable

Assume S is symmetric $\Rightarrow$

(1) all eigenvalues are real numbers

(2) There is an orthonormal set of eigen vectors.

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so $Q = [\vec{u}_1 \cdots \vec{u}_n]$ eigen vectors of S

$Q^{-1} S Q = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$. by eigenvalue theorem

$\parallel$

$Q^T S Q$ since Q is orthogonal matrix

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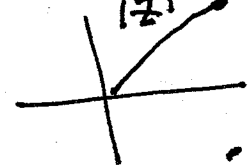
For the proof I need to look ahead to a future topic - complex vector spaces.

- $v \in \mathbb{C}^n$ means $v = (v_1, .., v_n)$ each $v_i \in \mathbb{C}$

we drop the arrow over v so we can take the conjugate



- $z = \alpha + i\beta \Rightarrow \bar{z} = \alpha - i\beta$
- $z\bar{z} = \alpha^2 + \beta^2 = |z|^2 \geq 0$
- $\bar{v} = (\bar{v}_1, .., \bar{v}_n)$

new definition → $\langle v, w \rangle = \bar{v}^T w = \bar{v}_1 w_1 + .. + \bar{v}_n w_n$

- $\langle v, v \rangle = \bar{v}^T v = \bar{v}_1 v_1 + .. + \bar{v}_n v_n$
  $= \sum |v_i|^2 = ||v||^2$
- $\langle \lambda v, w \rangle = \bar{\lambda} \langle v, w \rangle$, $\langle v, \lambda w \rangle = \lambda \langle v, w \rangle$

new definition

- $S^* = \bar{S}^T$ conjugate transpose or adjoint
- When S is real, $S^* = S \Leftrightarrow S^T = S$
- new definition When S is complex $S^* = S \Leftrightarrow S$ is Hermitian or self-adjoint
- So S is real then Hermitian $\Leftrightarrow$ symmetric
- $\langle u, Aw \rangle = \langle A^*u, w \rangle$ Proof

  $\overline{(A^*u)}^T w$

  $\langle A^*u, w \rangle =$

  $= \bar{u}^T \overline{(A^*)}^T w = \bar{u}^T (A^*)^* w = \bar{u}^T Aw = \langle u, Aw \rangle$

  $[A^{**} = A]$
- So S Hermitian $\Rightarrow$

  $\langle u, Aw \rangle = \langle Au, w \rangle$

  since $A^* = A$ justifies name of self-adjoint

Spectral Theorem part (1): If S is Hermitian (or Symmetric if it is real) then all its eigenvalues are real

say $Su = \lambda u$, $u \neq 0$, then

$$\langle \lambda u, u \rangle = \langle Su, u \rangle = \langle u, Su \rangle = \langle u, \lambda u \rangle = \lambda \langle u, u \rangle = \lambda \|u\|^2$$

$$\langle \lambda u, u \rangle = \bar{\lambda} \langle u, u \rangle = \bar{\lambda} \|u\|^2$$

Since $\|u\|^2 \neq 0$

$$\lambda = \bar{\lambda} \Rightarrow \underline{\lambda \text{ is real}}.$$

$$\alpha + i\beta = \alpha - i\beta$$

as required.