Spectral theorem $S^T = S$ symmetric $\Rightarrow$ 1

(1) eigenvalues are all real.
(2) there exists a collection of eigenvect. $\{\vec{u}_1, \ldots, \vec{u}_n\}$ that form an orthonormal basis.

so $U = [\vec{u}_1 \ldots \vec{u}_n] \Rightarrow U^T = U^{-1}$ (orthogonal)

Using the eigen-decomposition

$$U^{-1} S U = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$
$$\| $$
$$U^T S U$$

Proof of (1) last time. For (2) we are back in $\mathbb{R}^n$ since we know $\lambda_i \in \mathbb{R}$.

Assume now all eigenvalues are distinct

$\boxed{\text{New def:}}$ $\vec{w}^T$ is a left eigenvector with eigen value η if $\vec{w}^T A = \eta \vec{w}^T$

$\Rightarrow A^T \vec{w} = \eta \vec{w}$

- So left eigenvect of A are transposes of Right eigenvect of A^T

- So if $S^T = S \Rightarrow$ left and right eigenvect are transposes of each other.

- eigenvalues of a general A are the same as those of A^T. - RECALL, when A is square $\det(A^T) = \det(A)$

~~So~~ PROOF:

$$p_A(\lambda) = |A - \lambda I| = |(A - \lambda I)^T|$$
$$= |A^T - \lambda I^T| = |A^T - \lambda I| = p_{A^T}(\lambda)$$

But when A is not symmetric, its left and right eigenvectors ~~are~~ can be different.

FACT: If $(\vec{v}, \lambda)$ are eigenvect, val for A and $(\vec{w}^T, \eta)$ are left
eigenvect, eigenval
pair and $\lambda \neq \eta \Rightarrow \vec{w}^T \vec{v} = 0$

so $\vec{w} \perp \vec{v}$.

PROOF:

$$\eta (\vec{w}^T \cdot \vec{v}) = (\vec{w}^T A)\vec{v} = \vec{w}^T A \vec{v} = \vec{w}^T (A\vec{v}) = w^T \lambda \vec{v}$$
$$= \lambda (w^T \vec{v})$$

since $\eta \neq \lambda \Rightarrow \vec{w}^T v = 0$

i.e. $\vec{w} \perp \vec{v}$

<u>BACK TO The proof</u> of spectral th'm.: Since $S^T = S$, all its left eigenvect. are transposes of Right e.vect. $\Rightarrow$ If $\vec{v}_1$ and $\vec{v}_2$ are two right evect with $\lambda_1 \neq \lambda_2 \Rightarrow$

$$\vec{v}_1^T \vec{v}_2 = 0 \Rightarrow \vec{v}_1 \perp \vec{v}_2$$

So, if all eigenvalues are diff, all eigenvect are mutually orthogonal so

$\{\vec{v}_1, \dots, \vec{v}_n\}$ is an orthogonal set.

$\Rightarrow \left\{ \frac{\vec{v}_1}{\|\vec{v}_1\|}, \dots, \frac{\vec{v}_n}{\|\vec{v}_n\|} \right\}$ is o.n. set. of eigenvect.

New DEF: A symmetric matrix S is called postive definite (pos. def.) if all its eigenvalues are positive

so $U^T S U = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$ all $\lambda_i > 0$

using the spectral theorem

- post semi def is all $\lambda_i \geq 0$.
- neg def if all $\lambda_i < 0$

Three equiv def when S is Symmetric

(1) S has all posit eigenval (posit. def)

(2) S defines a positive energy

$Q(\vec{x}) = \vec{x}^T S x$ with

$Q(\vec{x}) > 0$ when $\vec{x} \neq 0$

(3) $S = A^T A$ for a matrix A with lin. ind. columns.

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note: In mechanics, the std Kinetic energy is $\frac{1}{2}\dot{x}^T M \dot{x}$ for sym. generalized mass matrix

<u>Proofs</u>: $(1) \Leftrightarrow (2)$. S has all post eigenvalues so $U^T S U = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$ with all $\lambda_i > 0$ write $S = U \Lambda U^T$

$$\vec{x}^T S \vec{x} = (\vec{x}^T U)\Lambda(U^T \vec{x}) = (U^T\vec{x})^T \Lambda (U^T \vec{x})$$

$$= \vec{y}^T \Lambda \vec{y} > 0$$

let $\vec{y} = U^T \vec{x}$

$$= \lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 > 0$$

when $\vec{y} \neq 0$ since all $\lambda_i > 0$

but $\vec{x} \neq 0 \Rightarrow \vec{y} \neq 0$ since U^T is invertible

Now assume $x^T S x > 0$ for all $\vec{x} \neq 0$.

once ~~again~~ write $\vec{y} = U^T \vec{x}$ and then

$$y^T \Lambda y = x^T S x > 0 \quad \text{for all } \underline{\underline{\vec{y}}}$$

$$\| $$

$$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2$$

let $\vec{y} = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} = \vec{e}_j$

$$\Rightarrow 0 < \vec{e}_j^T \Lambda e_j = \lambda_j$$

<u>Consequence</u> If S is post def Symm

$\Rightarrow$ define

$$\langle \vec{u}, \vec{v} \rangle_S = \vec{u}^T S \vec{v}$$

is an <u>inner product</u>.

(1) $\langle \vec{u}, \vec{u} \rangle_S \geq 0$ and $= 0$ only when $\vec{u} = 0$.

(2) $\langle \vec{u}, \vec{v} \rangle_S = \langle \vec{v}, \vec{u} \rangle_S$ since $S^T = S$

(3) $\langle \alpha \vec{u}, \vec{v} \rangle_S = \langle \vec{u}, \alpha \vec{v} \rangle_S = \alpha \langle \vec{u}, \vec{v} \rangle$

This ~~he~~ inner product adapted or derived from S.

2nd derivative test restated:

assume $\nabla \underline{\Phi}(x_0) = 0$

(1) $H\underline{\Phi}(x_0)$ pos def $\Rightarrow$ loc min

(2) $H\underline{\Phi}(x_0)$ neg def $\Rightarrow$ loc max

(3) $H\underline{\Phi}(x_0)$ is neither (indefinite)

$\Rightarrow$ saddle or no test

Next Time $S = A^T A$