LADS .14

S is symmetric if $S^T = S$ (square)

S is sym. post. def if all eigenvalues of S $\lambda_i > 0$ (1)

<u>Thm</u> S is sym. postive def. $\iff$ positive energy.

$Q(x) = \frac{1}{2} x^T S x$ with $Q(x) > 0$ when $x \neq 0$.

<u>Thm</u>: S is sym post def $\iff$ $S = \underline{\underline{A^T A}}$

with A having lin. ind. col.

(2)

($\Leftarrow$) $S = A^T A$ then

$2Q(x) = x^T S x = x^T A^T A x$

$= (Ax)^T Ax = \|Ax\|^2 \geq 0$

When is $A\cancel{x} = 0$? But if A is lin. ind. col $A = [\vec{c}_1 \ldots \vec{c}_n]$, $Ax = \sum x_i \vec{c}_i = 0$ only when $x_i = 0$ all i. So when A has lind col $\Rightarrow Ax = 0$ only when $x = 0$

So $2Q(x) = \|Ax\|^2 > 0$ so pos. def.

($\Rightarrow$) S is sym pos def all $\lambda_i > 0$.

By the spectral Th'm. $\Lambda = U^T S U$, $\underline{\underline{U^T = U^{-1}}}$

with all $\lambda_i > 0$, $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_n)$

or $\underline{S = U \Lambda U^T}$, let $\sqrt{\Lambda} = \text{diag}(\lambda_1^{1/2}, \dots, \lambda_n^{1/2})$

ok since $\lambda_i > 0$

Let $A = U \sqrt{\Lambda} U^T$

Then $A^T A = (U \sqrt{\Lambda} U^T)^T (U \sqrt{\Lambda} U^T)$

$= U (\sqrt{\Lambda})^T U^T U \sqrt{\Lambda} U^T$

$= U \sqrt{\Lambda} \sqrt{\Lambda} U^T$

$= U \Lambda U^T = S$ $\blacksquare$

Singular Value Decomposition

Geometric meaning

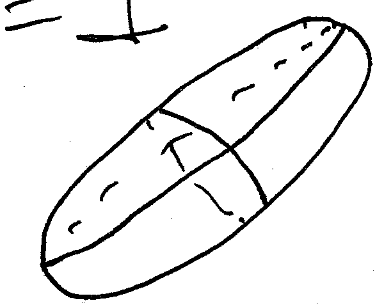
A linear transformation (mult. by a matrix)

Takes the unit sphere in $\mathbb{R}^n$

to an

$x_1^2 + \dots + x_n^2 = 1$

ellipsoid $\frac{y_1^2}{\sigma_1^2} + \dots + \frac{y_n^2}{\sigma_n^2} = 1$



Assume A is square then it has a decomposition

$$A = U \Sigma V^T$$

$$\Sigma = \mathrm{diag}(\sigma_1, .., \sigma_n), \quad \sigma_i \geq 0 \text{ all } i$$

U, V are $n \times n$ - orthogonal

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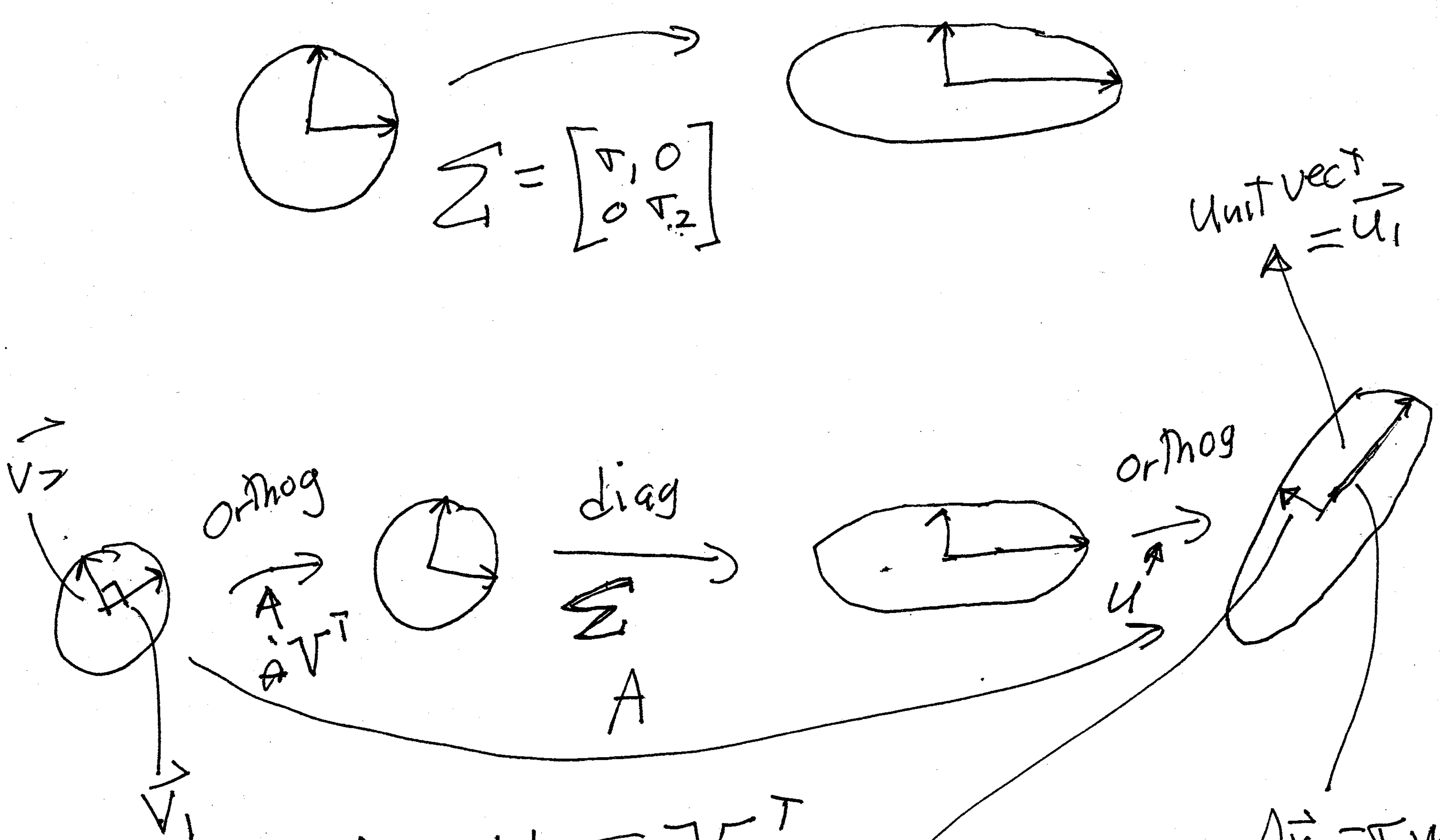
rewrite as $\quad AV = U\Sigma \qquad \sigma_i$ = singular values

$\vec{v}_j$ are the col of $V \leftarrow$ right sing vector

$\vec{u}_j$ are the col of $U \leftarrow$ left sing vect

$$\Rightarrow \quad A\vec{v}_j = \sigma_j \vec{u}_j \qquad \underline{\underline{\sigma_j \geq 0}}$$



$$A = U \Sigma V^T$$

$A\vec{v}_1 = \sigma_1 \vec{u}_1$

$\sigma_1 =$ length major axis

$A\vec{v}_2 = \sigma_2 \vec{v}_2$

<u>Proof:</u> <u>when A is square and invertible</u>

A^TA is sym, semi pos def

since A is invertible it is pos def

By the spectral Theorem

$$V^T(A^TA)V = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$$

all $\lambda_i > 0$.

Assume $V = [\vec{v}_1 \ldots \vec{v}_n]$, since V

is orthogonal, $\vec{v}_i^T \vec{v}_j = \delta_{ij}$

let $\sigma_i = \sqrt{\lambda_i}$

so $(A^T A)\vec{v}_i = \sigma_i^2 \vec{v}_i$ (evect eq for $A^T A$)

Let $\vec{u}_i = \dfrac{A\vec{v}_i}{\sigma_i}$ let $U = [\vec{u}_1 \ldots \vec{u}_n]$

(1) $A\vec{v}_i = \sigma_i \vec{u}_i$ by construction so

$$AV = U\Sigma$$ works automatically

with $\Sigma = \mathrm{diag}(\sigma_1, .., \sigma_n)$.

(2) We need that U is orthogonal or

$\vec{u}_i^T \vec{u}_j = \delta_{ij}$ which we check

$$u_i^T \vec{u}_j = \left(\frac{A\vec{v}_i}{\sigma_i}\right)^T \left(\frac{A\vec{v}_j}{\sigma_j}\right)$$

$$= \frac{\vec{v}_i^T (A^T A \vec{v}_j)}{\sigma_i \sigma_j} = \frac{\vec{v}_i^T \sigma_j^2 v_j}{\sigma_i \sigma_j}$$

$$= \underbrace{\frac{\sigma_j}{\sigma_i}}_{=1 \text{ when } i=j} \underbrace{\vec{v}_i^T v_j}_{\delta_{ij}} = \delta_{ij} \quad \text{as \underline{needed}}$$