Singular Value Decomposition, cont.

Simplest case. – A is $n \times n$, full rank. $\boxed{\text{LADS 15}}$

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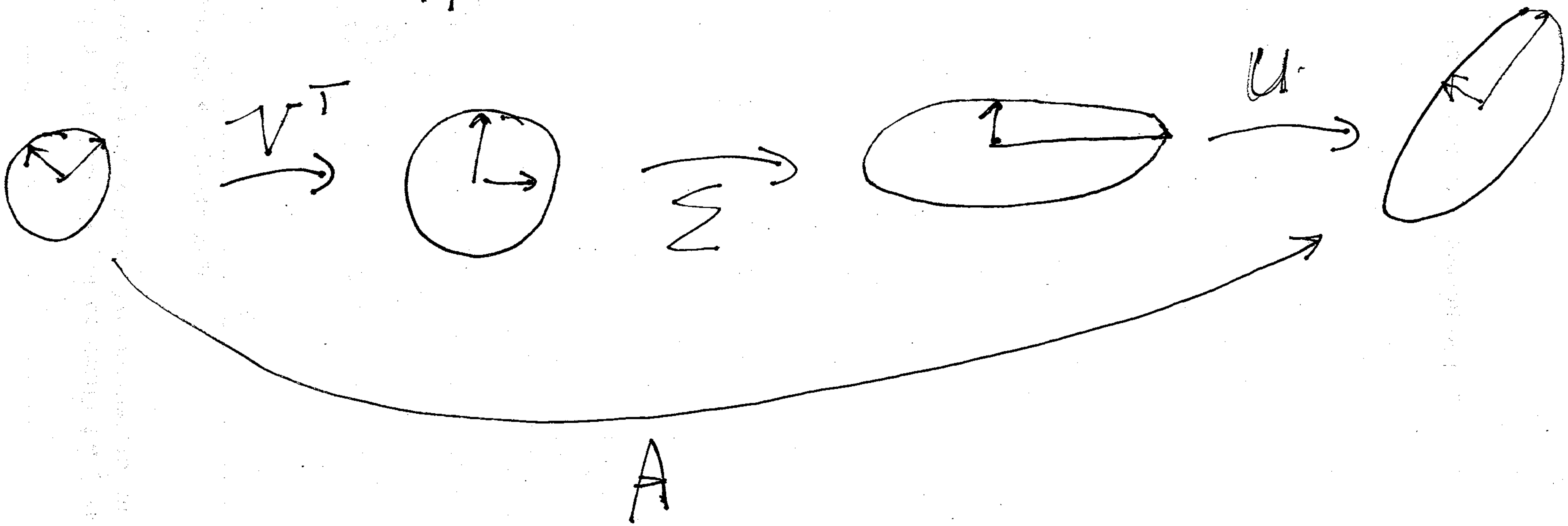
Then $A = U \Sigma V^T \Leftrightarrow AV = U\Sigma$

$\Sigma = \text{diag}(\sigma_1, \ldots, \sigma_n)$ $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$

U, V are $n \times n$ orthogonal.



$U = [\vec{u}_1 \dots \vec{u}_n] \quad V = [\vec{v}_1 \dots \vec{v}_n]$

$A = U\Sigma V^T \qquad AV = U\Sigma$

$\{\vec{u}_1, \dots, \vec{u}_n\}$ o.n basis

$\{\vec{v}_1, \dots, \vec{v}_n\}$ o.n. basis

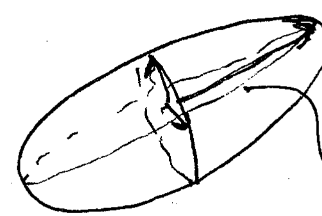
$A\vec{v}_j = \sigma_j \vec{u}_j$

$\vec{u}_j$ = left sing vect

$\vec{v}_j$ = right sing vect

$\vec{v}_j \xrightarrow{A} \sigma_i \vec{u}_j$

σ_i = singular values = lengths of axes of the image ellipsoid.



$\sigma_1 \vec{u}_1 = A\vec{v}_1$

major axis

Computing SVD by hand

Connecting the SVD to $\boxed{A^TA}$ and $\underline{\underline{AA^T}}$

Assume $A = U\Sigma V^T$

$$A^TA = V\Sigma^T U^T U \Sigma V^T$$
$$= V\Sigma^T\Sigma V^T \quad \leftarrow \text{assume } \Sigma \text{ is diagonal.}$$
$$= V\Sigma^2 V^{-1}$$
$$= V \operatorname{diag}(\sigma_1^2, \ldots, \sigma_n^2) V^{-1}$$

So A^TA is similar to Σ^2 so: eigenvalues of A^TA are $\{\sigma_1^2, \ldots, \sigma_n^2\}$

OR if the eigenvalues of A^TA are $\lambda_1, \ldots, \lambda_n$

$\Rightarrow$ sing values of A are $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_n}$

and $V^{-1}(A^TA)V = \Sigma^2$.

So $\underline{\underline{\vec{v}_1, \ldots, \vec{v}_n}}$ are the eigenvectors of A^TA

right sing vect.

Now consider AA^T

$= U\Sigma V^T V \Sigma^T U^T$ & $\boxed{\Sigma \text{ is square}}$

$= U\Sigma^2 U^{-1}$

$\Rightarrow$ σ_i^2 are the evalues of AA^T

and the $\underline{\underline{\vec{u}_i}}$ are the eigenvect of AA^T.

Compute (by hand) the SVD of

$$A = \begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} \qquad \begin{bmatrix} 3 & 4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} 25 & 2 \\ 20 & 2 \end{bmatrix}$$

(1) Form $A^T A = \begin{bmatrix} 25 & 20 \\ 20 & 25 \end{bmatrix}$

(2) Find the eigen val + eigen vect of $A^T A$

$$\lambda_1 = 45 \quad \vec{W_1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 5 \quad \vec{W_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

not unit vectors.

(3) $\sigma_2 = \sqrt{5} \qquad \sigma_1 = \sqrt{45} = 3\sqrt{5}$

largest singular value is \sigma_1

(4) normalize the $\vec{v}_i$:

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix} = \frac{\vec{w}_1}{\|\vec{w}_1\|}$$

$$\vec{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\1\end{bmatrix} \qquad V = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & -1\\1 & 1\end{bmatrix}$$

(5) Since $A\vec{v}_i = \sigma_i \vec{u}_i \Rightarrow$

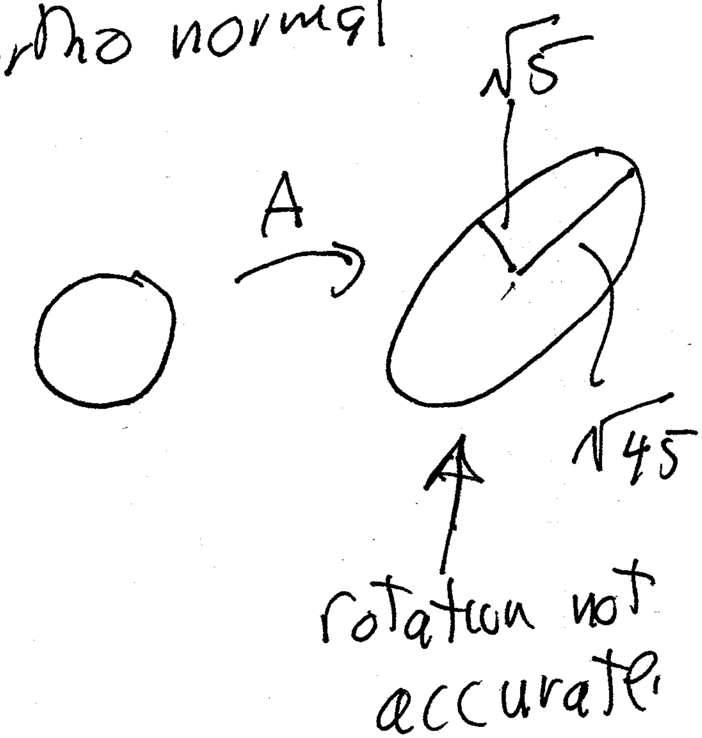
$$\vec{u}_i = \frac{A\vec{v}_i}{\sigma_i} \quad (\sigma_i \neq 0)$$

$$\vec{u}_1 = \frac{A\vec{v}_1}{\sigma_1} = \frac{\frac{3}{\sqrt{2}}\begin{bmatrix}1\\3\end{bmatrix}}{\sqrt{45}} = \frac{1}{\sqrt{10}}\begin{bmatrix}1\\3\end{bmatrix}$$

$$\vec{u}_2 = \frac{A\vec{v}_2}{\sigma_2} = \frac{\frac{1}{\sqrt{2}}\begin{bmatrix}-3\\1\end{bmatrix}}{\sqrt{5}} = \frac{1}{\sqrt{10}}\begin{bmatrix}-3\\1\end{bmatrix}$$

These are automatically ortho normal

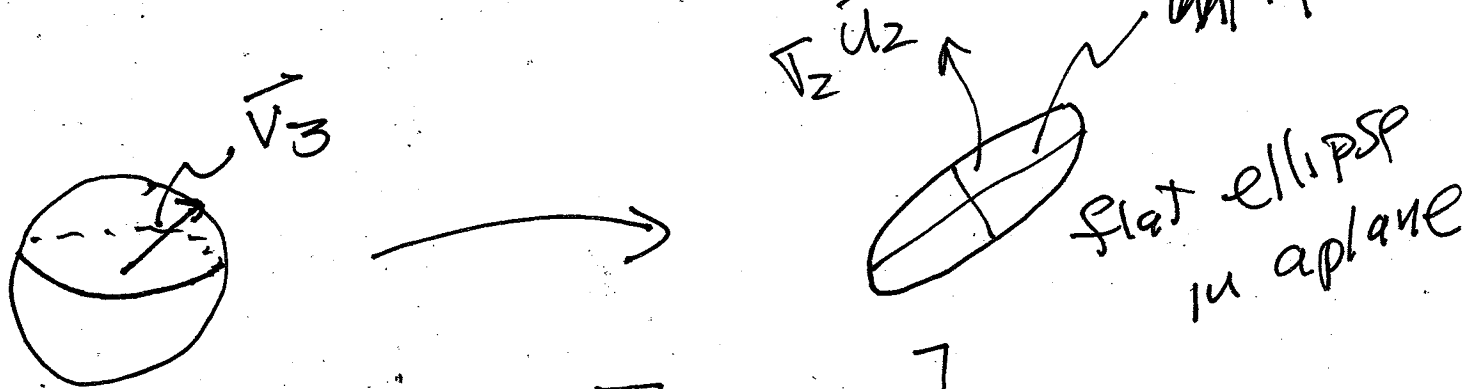
$$\underline{U} = \begin{bmatrix} 1/\sqrt{10} & -3/\sqrt{10} \\ 3/\sqrt{10} & 1/\sqrt{10} \end{bmatrix}$$



$$\underline{\Sigma} = \begin{bmatrix} \sqrt{45} & 0 \\ 0 & \sqrt{5} \end{bmatrix}$$

Additional cases

A is not full rank



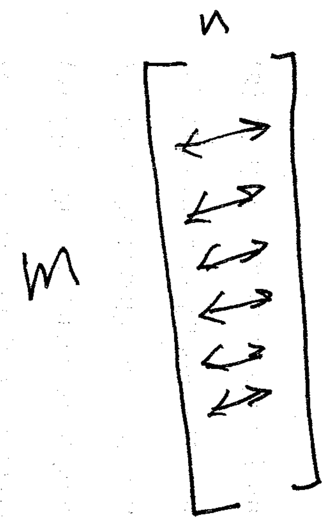
$$\Sigma = \begin{bmatrix} \sigma_1 & & 0 \\ & \sigma_2 & \\ 0 & & 0 \end{bmatrix}$$

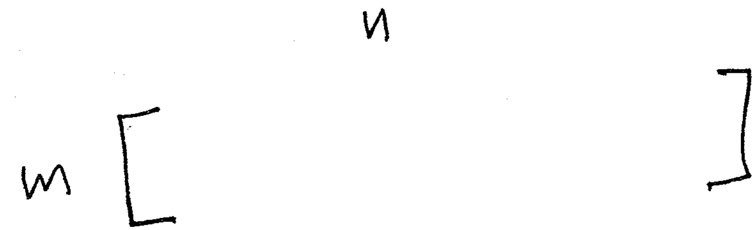
$A\vec{v}_3 = 0 \cdot \vec{u}_3 = \vec{0}$

$\text{Null}(A) = \text{span}(\vec{v}_3)$

$\text{rank}(A) = 2$ rowspace$(A) = \text{span}(\vec{u}_1, \vec{u}_2)$

other cases A is not square.





$m > n$

$\underline{\underline{m < n}}$

Now A^TA and AA^T are different dimensions $m \neq n$

but their non zero eigen values are still the same. and sing values of A are $\sigma_i = \sqrt{\lambda_i}$

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What does the SVD look like.

$$\underbrace{\begin{bmatrix} \end{bmatrix}}_{m \times n} = \underbrace{\begin{bmatrix} \text{orthog} \end{bmatrix}}_{m \times m} \underbrace{\begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \\ & & & \sigma_n \\ \hline & 0 & \end{bmatrix}}_{m \times n} \underbrace{\begin{bmatrix} \text{orthog} \end{bmatrix}}_{n \times n}$$

$m > n$

generalized diagonal.

$m < n$.

$$\begin{array}{c c} & n \\ m & \begin{bmatrix} & & \end{bmatrix} \end{array} = \begin{array}{c c} & m \\ m & \begin{bmatrix} \text{orthog} \end{bmatrix} \end{array} \left[\begin{array}{ccc|c} \sigma_1 & & & \\ & \ddots & & 0 \\ & & \sigma_m & \end{array} \right] \begin{bmatrix} \\ \text{orthog} \\ \\ \end{bmatrix}$$

generalized diagonal!

In both cases
# of sing values not zero is at most $\min(m, n)$.

Full SVD

$$A = U \underset{m \times n}{\Sigma} V^T$$

A is $m \times n$

V is $n \times n$ orthogonal

U is $m \times m$ orthogonal.

Σ is $m \times n$ generalized diagonal