$\mathcal{B}$ is a $\boxed{\text{basis}}$ $\mathcal{B} = \{\vec{v}_1, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ if every $\vec{w} \in \mathbb{R}^n$ can be written in a unique way as

$$\vec{w} = \alpha_1 \vec{v}_1 + \dots + \alpha_n \vec{v}_n$$

α_i are the <u>coordinates</u> w.r.t. the basis.

<u>FACT</u>: Every basis for $\mathbb{R}^n$ has exactly <u>n</u> vectors, <u>n = dimension</u>

<u>Theorem</u>: $\mathcal{B}$ is a basis if and only if

(1) $\mathcal{B}$ <u>spans</u> $\mathbb{R}^n$, i.e. any $\vec{w} \in \mathbb{R}^n$ can be written as

$$\vec{w} = \sum_{i=1}^{n} \alpha_i \vec{v_i}, \quad \boxed{\operatorname{span}(\mathcal{B}) = \mathbb{R}^n}$$

(2) $\mathcal{B}$ is <u>linearly independent</u> if $\sum_{i=1}^{n} \alpha_i \vec{v_i} = \vec{0}$ implies $\alpha_i = 0$ all i.

<u>Proof</u> ($\Rightarrow$) <u>easy</u> ($\Leftarrow$)

By (1), every $\vec{w}$ can be written in some fashion as

$$\vec{w} = \sum \alpha_i \vec{v_i}.$$

We need this to be unique. Assume <u>not</u>

$$\vec{w} = \sum \alpha_i \vec{v_i} = \sum \beta_i \vec{v_i}$$

some $\alpha_i \neq \beta_i$

Then $\vec{0} = \sum (\alpha_i - \beta_i) \vec{v_i}$

with some $\alpha_i - \beta_i \neq 0$

Contradicts assumption of (2).

n = big

$\mathbb{R}^n$ all $\vec{v} = \begin{bmatrix} v_1 \\ \vdots \\ w_n \end{bmatrix}$

$\vec{w}^T = [w_1 \; \cdots \; w_n]$ T = Transpose.

$$\vec{w}^T v = \sum_{j=1}^{n} w_i v_i = \vec{w} \cdot \vec{v}$$

$\vec{w} \perp \vec{v}$ perpendicular, <u>orthogonal</u> if

$$\vec{w} \cdot \vec{v} = 0$$

properties of dot product

$$(\vec{v}+\vec{u})\cdot\vec{w} = \vec{v}\cdot\vec{w} + \vec{u}\cdot\vec{w}$$

$$\vec{u}\cdot\vec{w} = \vec{w}\cdot\vec{u}$$

$$(\alpha\vec{u})\cdot\vec{w} = \alpha(\vec{u}\cdot\vec{w})$$

Linear combination

$$\sum_{i=1}^{n} \alpha_i \vec{w_i}$$

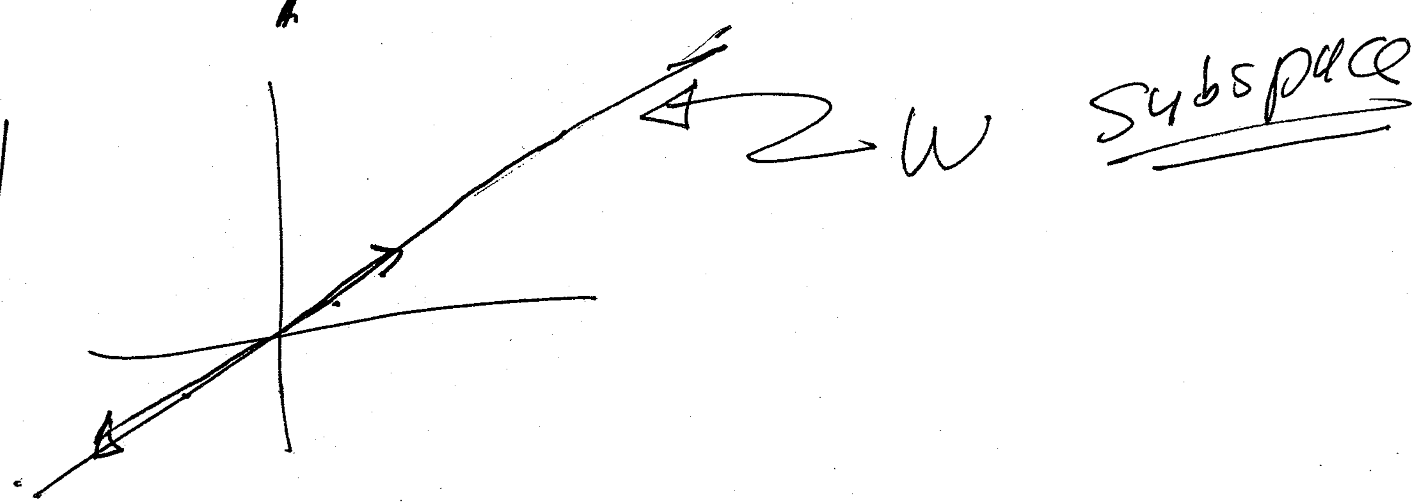
__Subspaces__ - subcollections of data where we can do linear algebra

$W \subseteq \mathbb{R}^n$ is a __subspace__ if it is closed under linear combinations.

__if__ $\vec{w}_1 \in W$, $\vec{w}_2 \in W \Longrightarrow$

$\alpha_1 \vec{w}_1 + \alpha_2 \vec{w}_2 \in W$ This implies $\vec{0} \in W$

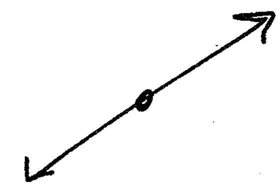
eg||



Examples in $\mathbb{R}^3$

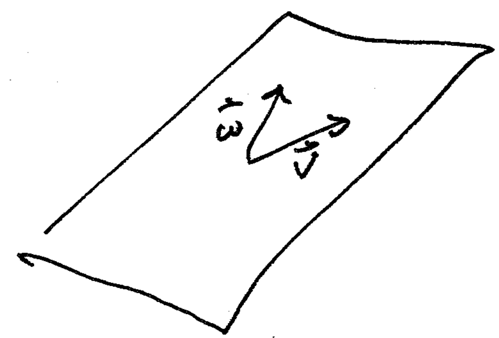
(0) $\{\vec{0}\}$, just the origin

(1) Lines through origin $\{\alpha \vec{v}\}$



(2) planes through origin

$\{\alpha_1 \vec{v} + \alpha_2 \vec{w}\}$



(3) All of $\mathbb{R}^3$

Subspaces have dimension 0, 1, 2 or 3

Examples in $\mathbb{R}^n$

(1) Any collection of vectors $\vec{w}_1, \ldots, \vec{w}_k$. Let W be all linear combinations of the $\vec{w}_i$.

$$\sum_{i=1}^{k} \alpha_i \vec{w}_i$$

Why is W a subspace?

$\vec{u}_1, \vec{u}_2 \in W$ we need to show $\alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 \in W$

Since $\vec{u}_1, \vec{u}_2 \in W$, By definition

$$\vec{u}_1 = \sum \beta_i \vec{w}_i$$ (supressing summation indices as understood)

$$\vec{u}_2 = \sum \gamma_i \vec{w}_i$$

$$\Rightarrow \quad \alpha_1 \vec{u}_1 + \alpha_2 \vec{u}_2 = \alpha_1 \sum \beta_i \vec{w}_i + \alpha_2 \sum \gamma_i \vec{w}_i$$

$$= \sum (\alpha_1 \beta_i + \alpha_2 \gamma_i) \vec{w}_i$$ (distribute and collect)

$$= \sum \sigma_i \vec{w}_i \in W$$

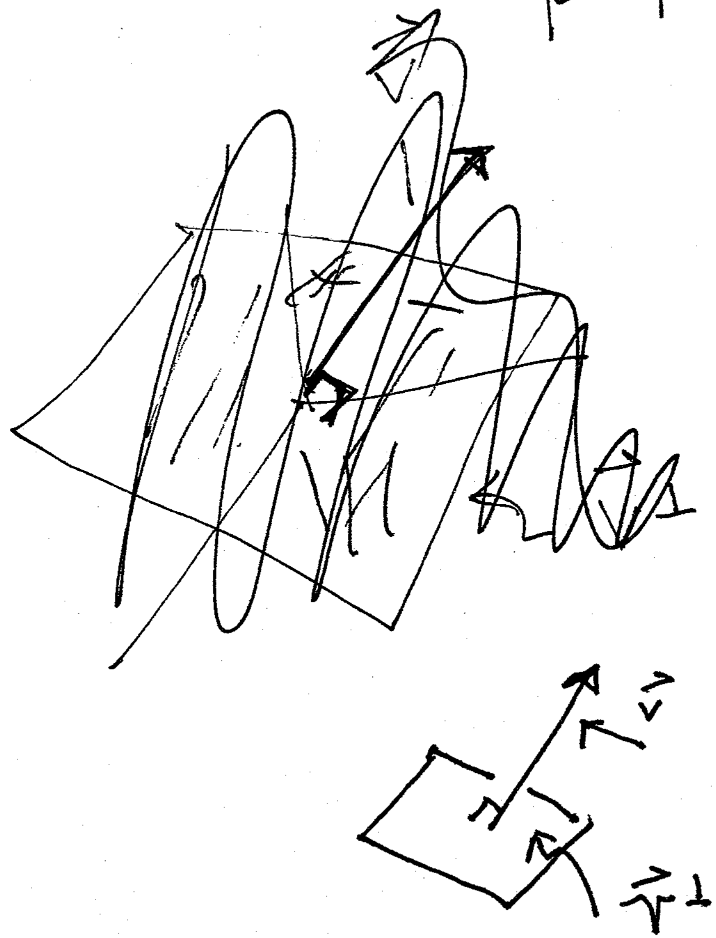
where $\sigma_i = \alpha_1 \beta_i + \alpha_2 \gamma_i$

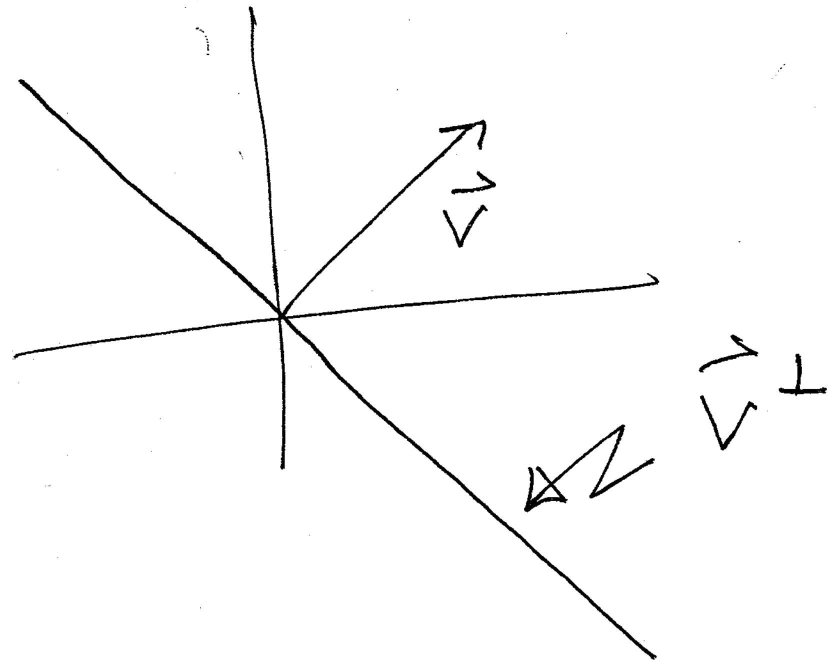
(2) $\vec{v} \in W$

Define $\vec{v}^{\perp} = \{\vec{u} : \vec{u} \perp \vec{v}\}$

perp

$=$ orthogonal subspace to $\vec{v}$





$\vec{V}^{\perp}$ is a subspace : Proof:

$\vec{u}, \vec{w} \in \vec{V}^{\perp}$ so $\vec{u} \cdot \vec{V} = 0$,

$\vec{w} \cdot \vec{V} = 0$, $(\alpha_1 \vec{u} + \alpha_2 \vec{w}) \cdot \vec{V}$

$= \alpha_1 \vec{u} \cdot \vec{V} + \alpha_2 \vec{w} \cdot \vec{V} = \alpha_1 \cdot 0 + \alpha_2 \cdot 0 = 0$ using properties of dot product

so $\alpha_1 \vec{u} + \alpha_2 \vec{w} \in \vec{V}^{\perp}$

$\boxed{\text{matrices or arrays}}$

A is $m \times n$ (rows $\times$ columns)

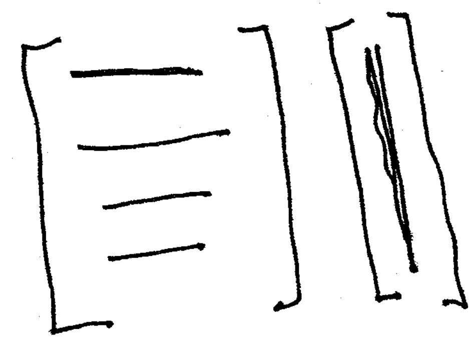
$$A = \begin{array}{cc} & j \\ i & \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & \boxed{A_{ij}} & \\ A_{m1} & \cdots & & A_{mn} \end{bmatrix} \end{array} = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$$

$A = (A_{ij})$ $\quad A_{ij}$ = entry in i^{th} row
j^{th} column

matrix times vector

$A\vec{x}$ $\quad$ $\left.\begin{array}{l} A \text{ is } m \times \underline{\underline{n}} \\ \vec{x} \text{ is } \underline{\underline{n}} \times 1 \end{array}\right\} A\vec{x} = m \times 1$

$$A\vec{x} = \begin{bmatrix} \sum_{j=1}^{n} A_{1j} X_j \\ \sum_{j=1}^{n} A_{2j} X_j \\ \vdots \\ \sum_{j=1}^{n} A_{mj} X_i \end{bmatrix}$$



$A = \begin{bmatrix} \vec{r}_1 \\ \vdots \\ \vec{r}_m \end{bmatrix}$ $\quad$ $A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$