Brief essentials of Numerical Linear Algebra

Conditioning: error propagation and
Stability: magnification

Conditioning: measure the response to perturbations or errors of the mathematical problem.

Stability measure the response to perturbation or errors of an algorithm or its implementation on a computer.

$\boxed{\text{prob}} \longrightarrow \boxed{\text{alg}} \longrightarrow \boxed{\text{imp}} \longrightarrow \boxed{\text{hardware}}$

conditioning in general.

$$\mathbb{X} \xrightarrow{f} \mathbb{Y}$$

inputs data — "problem" — outputs solutions.

$(b, A) \longrightarrow x$ soln to $Ax = b$

(polynomial) $\longrightarrow$ Roots of poly.

If you vary b, how does x vary.

$$\mathbb{X} \xrightarrow{f} \mathbb{Y}$$

$x \rightsquigarrow x + \delta x$ perturbation error δx

$f(x) \rightsquigarrow f(x+\delta x) - f(x) = \delta f$

Roughly, condition # (absolute)

$$\frac{\delta f}{\delta x}$$

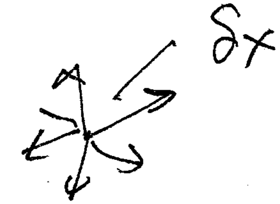
change in soln $\approx$ $\underbrace{\left(\text{condition number}\right)}_{\text{error magnification}} \times \left(\text{change in input}\right)$

Large condition # $\Longleftrightarrow$ Ill conditioned

Small " " $\Longleftrightarrow$ well conditioned

More formally, The abs. cond. #.

$$\hat{\kappa}(x) = \lim_{\delta \to 0} \max_{\Delta} \left\| \frac{\delta f}{\delta x} \right\|$$



$$\|\delta x\| < \delta$$

Think of δx infinitesimal

Relative condition

$$K(x) = \max_{\delta x} \frac{\dfrac{\|\delta f\|}{\|f(x)\|}}{\dfrac{\|\delta x\|}{\|x\|}}$$

$$= \max_{\delta x} \frac{\|\delta f\|}{\|\delta x\|} \bigg/ \frac{\|f(x)\|}{\|x\|}.$$

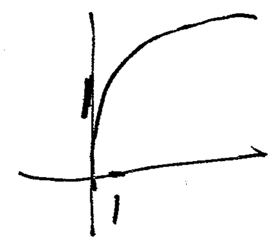
$$= \|Df\| \Big/ \|f(x)\| / \|x\|.$$

$\hat{K}(x) = \|Df(x)\|$ in some norm.

e.g. $f(x) = \sqrt{x}$ $x > 0$

$f'(x) = \frac{1}{2\sqrt{x}}$ $\hat{K}(x) = \left|\frac{1}{2\sqrt{x}}\right|$

large with x close to zero.



$K(x) = \left|\frac{\frac{1}{2\sqrt{x}}}{\frac{\sqrt{x}}{x}}\right| = 1/2$ well conditioned

Famous example: Wilkinson polynomial.

$$P(x) = \prod_{i=1}^{20} (x-i) \quad \text{degree 20 polyn.}$$

let a_{15} = the coeff of x^{15}

$x_{15} = 15$, unperturbed root

$$a_{15} + \delta a \longrightarrow \text{Root } \boxed{x_5 + 10^{13}\delta a.}$$

$$\boxed{K = 10^{13}}$$

Condition # of matrix mult + linear equation solving

1st hint: $A = U \Sigma V^T$

Find sensistivity of perb in b to soln x to $Ax = b$. assume A is invertible

of $\vec{x} = A^{-1}\vec{b}$

and $A^{-1}(\vec{b} + \delta\vec{b}) = A^{-1}b + \boxed{A^{-1}\delta b}$

orig soln   change in soln

$$A^{-1}\vec{\delta b} = \left(V \begin{bmatrix} 1/\sigma_1 & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sigma_n} \end{bmatrix} U^T \right) \vec{\delta b}$$

$=$

$$= \left(\frac{1}{\sigma_1} \vec{v}_1 \vec{u}_1^T + \cdots \frac{1}{\sigma_n} \vec{v}_n \vec{u}_n^T \right) \delta \vec{b}$$

σ_n is smallest sing value.

if σ_n is small,

$\frac{1}{\sigma_n}$ is <u>large</u>

so perb to soln is large.

Th'm (1) The condition κ under perb in $\vec{b}$ of the soln x to $Ax = \vec{b}$ is (2-norm).

$$\kappa_2 = \|A\|_2 \|A^{-1}\|_2$$

$$= \sigma_1 \frac{1}{\sigma_n} = \boxed{\frac{\sigma_1}{\sigma_n}}$$

(2) (1) follows from: the condition of Ax to perb of x is also

$$\kappa = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$$

So $\kappa_2 = \|A\|_2 \|A^{-1}\|_2 = \frac{\sigma_1}{\sigma_n}$ is the condition number of the square, invertible A.

Proof of (2)! $f(x) = Ax$

Note: all norms are 2-norms

$$K(x) = \sup_{\delta x} \frac{\|\delta f\|}{\|f\|} \Big/ \frac{\|\delta x\|}{\|x\|}$$

$$= \sup_{\delta x \neq 0} \frac{\|A(x+\delta x) - A(x)\|}{\|Ax\|} \Big/ \frac{\|\delta x\|}{\|x\|}$$

$$= \sup_{\delta x \neq 0} \boxed{\frac{\|A\delta x\|}{\|\delta x\|}} \cdot \frac{\|x\|}{\|Ax\|}.$$

$$= \|A\| \sup \frac{\|x\|}{\|Ax\|}.$$

def of norm

smallest singular value

$\left[\text{let } \vec{x} = v_n, \quad Ax = \sigma_n u_n \right]$

achieves sup

MAX

13

$$= \|A\| \cdot \frac{\|\vec{v}_n\|}{\|\sigma_n \vec{u}_n\|}$$

v_n and u_n are unit vectors

$$= \|A\| \frac{1}{\sigma_n} = \|A\|_2 \|A^{-1}\|_2.$$

For part (1) Notice. $x = A^{-1} b$ is the soln, so it turns into (2) using A^{-1} and b so

$$K = \|A^{-1}\| \, \|(A^{-1})^{-1}\| = \|A^{-1}\| \, \|A\|$$

$$= K_2(A)$$

General Rule: If $K_2(A)$ is 10^d

$\Rightarrow$ soln will lose d digits of significance.

Hilbert MATRIX $\quad H_{ij} = \dfrac{1}{i+j-1}$

for $i = 1, \ldots, n$, $j = 1, \ldots, n$ $\quad \begin{bmatrix} 1 & 1/2 & \\ 1/2 & \cdots & \\ & & \end{bmatrix}$

$n = 5$, $K(H) \approx 5 \times 10^5$