QR Decomp (Factorization).

$$B = \begin{array}{cc} & \\ m & \begin{bmatrix} \vec{b}_1 & \cdots & \vec{b}_n \\ & & \end{bmatrix} \\ & n \end{array} \qquad \text{rank}(B) = n$$

Goal: Produce o.n. set $\{\vec{q}_1, \ldots, \vec{q}_n\}$ with $\{\vec{q}_1, \ldots, \vec{q}_k\}$ o.n. basis for $\text{span}(\vec{b}_1, \ldots, \vec{b}_k)$

The $k^{\underline{th}}$ step in Gram-Schmidt

$$\vec{v}_k = \vec{b}_k - (\vec{b}_k \cdot \vec{q}_1)\vec{q}_1 + \cdots - (\vec{b}_k \cdot \vec{q}_{k-1})\vec{q}_{k-1}$$

$$\vec{q}_k = \frac{\vec{v}_k}{\|v_k\|_2}$$

This yields Q and

$QR = A$ so $R = Q^T A$

since $Q^T Q = \underline{I}$

eg!

$$B = \begin{bmatrix} -1 & -1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

eq2

$$\vec{b}_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \quad \vec{q}_1 = \frac{\vec{b}_1}{\|\vec{b}_1\|} = \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{V}_2 = \vec{b}_2 - (\vec{b}_2 \cdot \vec{q}_1)\vec{q}_1 = \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - \left(\begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} \right) \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - \underbrace{\left[1/2 + 3/2 + 1/2 + 3/2 \right]}_{4} \begin{bmatrix} -1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ -1 \\ 3 \end{bmatrix} - \begin{bmatrix} -2 \\ 2 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\vec{q}_2 = \frac{\vec{V}_2}{\|\vec{V}_2\|} = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

eq3

$$\vec{v}_3 = \vec{b}_3 - (\vec{b}_3 \cdot \vec{q}_1)\vec{q}_1 - (\vec{b}_3 \cdot \vec{q}_2)\vec{q}_2$$

$$= \begin{bmatrix}1\\3\\5\\7\end{bmatrix} - \left(\begin{bmatrix}1\\3\\5\\7\end{bmatrix} \cdot \begin{bmatrix}-1/2\\1/2\\-1/2\\1/2\end{bmatrix}\right)\begin{bmatrix}-1/2\\1/2\\-1/2\\1/2\end{bmatrix} - \left(\begin{bmatrix}1\\3\\5\\7\end{bmatrix} \cdot \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}\right)\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$= \begin{bmatrix}1\\3\\5\\7\end{bmatrix} - \underbrace{\left(-1/2 + 3/2 - 5/2 + 7/2\right)}_{2}\begin{bmatrix}-1/2\\1/2\\-1/2\\1/2\end{bmatrix} - \underbrace{\left(\tfrac{1}{2} + \tfrac{3}{2} + \tfrac{5}{2} + \tfrac{7}{2}\right)}_{8}\begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$= \begin{bmatrix}1\\3\\5\\7\end{bmatrix} - \begin{bmatrix}-1\\1\\-1\\1\end{bmatrix} - \begin{bmatrix}4\\4\\4\\4\end{bmatrix} = \begin{bmatrix}-2\\-2\\2\\2\end{bmatrix}$$

$$q_3 = \frac{\vec{v}_3}{\|v_3\|} = \frac{1}{\sqrt{16}}\begin{bmatrix}-2\\-2\\2\\2\end{bmatrix} = \begin{bmatrix}-1/2\\-1/2\\1/2\\1/2\end{bmatrix}$$

$$\text{so } Q = \begin{bmatrix}-1/2 & 1/2 & -1/2\\ 1/2 & 1/2 & -1/2\\ -1/2 & 1/2 & 1/2\\ 1/2 & 1/2 & 1/2\end{bmatrix}$$

Q has orthonormal columns.

$\boxed{eg\,4}$

$B = QR \Rightarrow Q^T B = R$ so

$$R = \begin{bmatrix} -1/2 & 1/2 & -1/2 & 1/2 \\ 1/2 & 1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 \\ 1 & 3 & 3 \\ -1 & -1 & 5 \\ 1 & 3 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & 2 \\ 0 & 2 & 8 \\ 0 & 0 & 4 \end{bmatrix}$$

This is the thin or reduced QR.

often written $A = \hat{Q}\hat{R}$

$$\begin{bmatrix} A \end{bmatrix} = \underbrace{\begin{bmatrix} \hat{Q} \end{bmatrix}}_{(m \times n)} \underbrace{\begin{bmatrix} \hat{R} \end{bmatrix}}_{(n \times n)}$$

Full QR

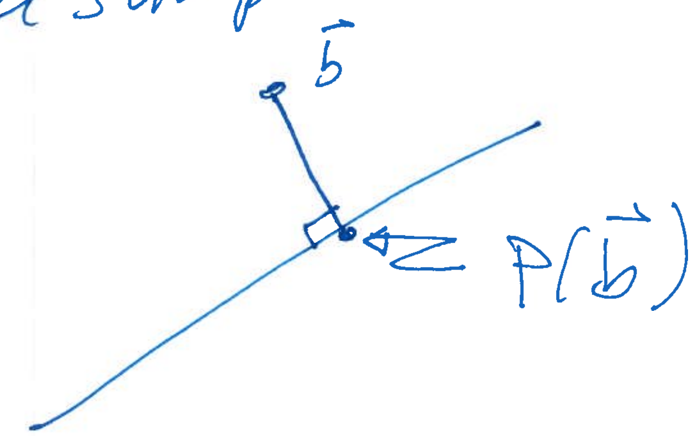
new o.n. col to complete the basis

$$\begin{bmatrix} A \end{bmatrix} = {}_{m}\overset{m}{\underbrace{\begin{bmatrix} \hat{Q} & | & | & | \end{bmatrix}}_{m \times m}} \underbrace{\begin{bmatrix} \hat{R} \\ 0 \end{bmatrix}}_{m \times n}$$

Q is orthogonal

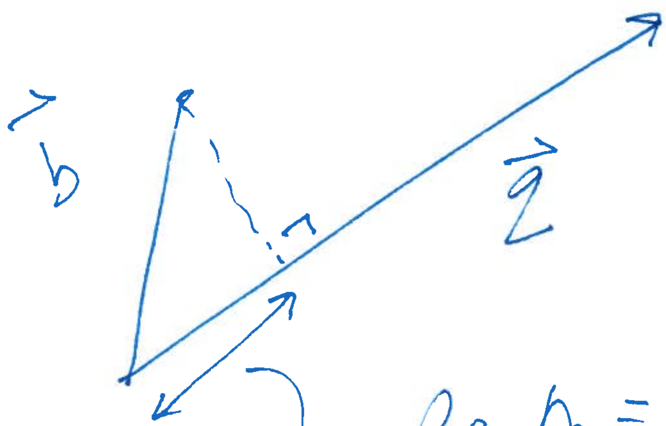
$\boxed{\text{QR and orhogonal projection}}$

Imp. in optimization since <u>Shortest</u> (min). distance from a pt. to a subspace is orhogonal projection



What is the formula for $P =$ orhogonal projection?

One dim!



$\vec{q}$ is unit vector

length $= \vec{b}\cdot\vec{q}$

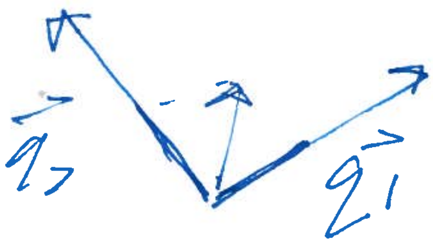
vector is $(\vec{b}\cdot\vec{q})\vec{q}$

$$(\vec{b}\cdot\vec{q})\vec{q} = \vec{q}(\vec{q}\cdot\vec{b}) = \vec{q}(\vec{q}^T\vec{b})$$

$$= \underbrace{(\vec{q}\vec{q}^T)}_{\text{outer product} \,=\, \text{rank 1 matrix}}\vec{b}$$

So $P(\vec{b}) = (\vec{q}\vec{q}^T)\vec{b}$ so $P = \vec{q}\vec{q}^T$

<u>Now projection onto a plane</u>



$$\vec{p}(\vec{b}) = (\vec{b}\cdot\vec{q}_1)\vec{q}_1 + (\vec{b}\cdot\vec{q}_2)\vec{q}_2.$$

$$= (q_1 q_1^T)\vec{b} + (\vec{q}_2\cdot q_2^T)\vec{b}$$

$$= \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \end{bmatrix} \vec{b}$$

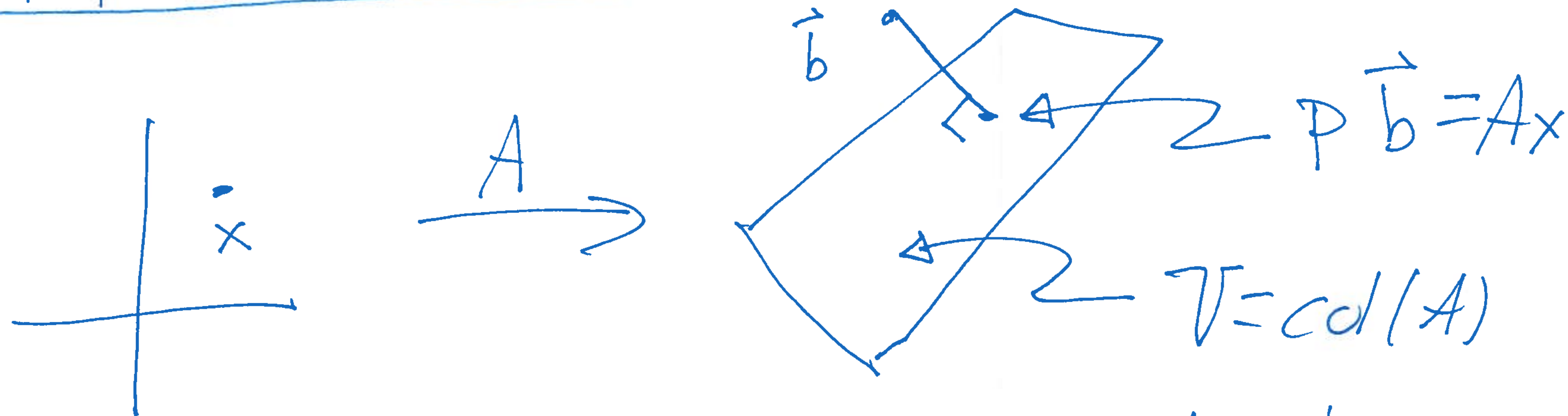
So $P = QQ^T$ with $Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix}$

__Theorem:__ If V has o.n. basis $\{\vec{q}_1, \ldots, \vec{q}_k\}$ let $\hat{Q} = \begin{bmatrix} \vec{q}_1 \cdots \vec{q}_k \end{bmatrix}$

Then orthogonal projection onto V is given by $P = \hat{Q}\hat{Q}^T$

or $P\vec{b} = (\hat{Q}\hat{Q}^T)\vec{b}$

Application to least squares



$Ax = b$

The soln to least sq

is $Ax = P\vec{b}$.

Now say $A = \hat{Q}\hat{R}$ factorization

so $\hat{Q}$ has columns that are an o.n.

basis for $col(A)$.

So projection onto col(A) is given by $P = \hat{Q}\hat{Q}^T$

$$Ax = P\vec{b} \quad \text{becomes}$$

$$\hat{Q}\hat{R}x = \hat{Q}\hat{Q}^T\vec{b}$$

$$\boxed{\hat{R}x = \hat{Q}^T\vec{b}} \quad \text{since } \hat{Q}^T\hat{Q} = I$$

easy to solve by back substitution.