MATRIX VECTOR Products

$$A = \begin{array}{cc} & n \\ m & \begin{bmatrix} \quad \end{bmatrix} \end{array} \qquad m \times n \qquad A = (A_{ij})$$

$$i = 1, \ldots, m$$

$$j = 1, \ldots, n$$

$$\vec{x} = \begin{bmatrix} \quad \end{bmatrix} \qquad n \times 1$$

$$A\vec{x} = (m \times \cancel{n}) \cdot (\cancel{n} \times 1) = m \times 1$$

$$\vec{x} \longmapsto A\vec{x}$$

$\vec{x}$ = input

$A\vec{x}$ = output

$$\vec{x} \rightarrow \boxed{} \rightarrow A\vec{x}$$

Reminder $\vec{v}$

$\vec{u}, \vec{v}$ column vect

$$\boxed{\vec{u}^T \vec{v}} = [u_1 \dots u_n] \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} = \boxed{\vec{u} \cdot \vec{v}} = \langle \vec{u}, \vec{v} \rangle$$

Different ways of viewing matrix vector products

$$A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vec{r}_2^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \qquad A\vec{x} = \begin{bmatrix} \vec{r}_1^{\,T}\vec{x} \\ \vec{r}_2^{\,T}\vec{x} \\ \vdots \\ \vec{r}_m^{\,T}\vec{x} \end{bmatrix} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix}$$

$$A\vec{x} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 - 2 + 6 \\ 0 \;\; 1 + 4 \\ 3 - 2 \;\; 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

MatRix vector products are linear

$$A\left(\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2\right)$$
$$= \alpha_1 A(\vec{x}_1) + \alpha_2 A(\vec{x}_2)$$

Various subspaces associated with a matRix

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① Null space OR kernal

$$\mathrm{Null}(A) = \{\vec{x} : A\vec{x} = \vec{0}\}$$

is a subspace.

Proof $\vec{x}_1, \vec{x}_2 \in \text{Null}(A)$

$$A\vec{x}_1 = \vec{0} \qquad A\vec{x}_2 = \vec{0}$$

$$A(\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2) = \alpha_1 A\vec{x}_1 + \alpha_2 A\vec{x}_2$$
$$= \vec{0} + \vec{0} = \vec{0}$$

So $\alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2 \in \text{Null}(A)$

So Null(A) is closed under scalar multiplication and so is a subspace. ∎

Another interpretation of Null(A). Involves 2nd important subspace

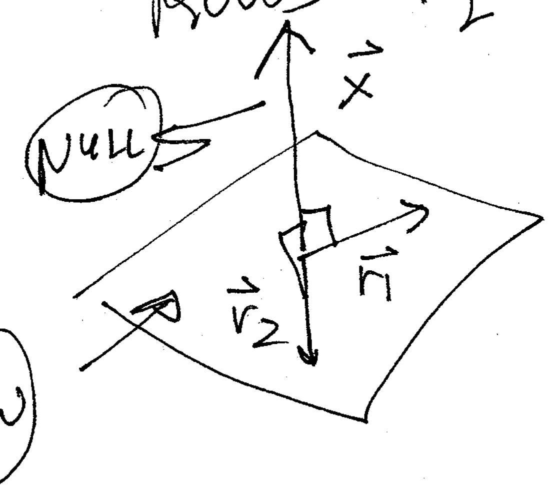
$$A = \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix}$$

Row(A) = Span($\vec{r}_1^{\,T}, \dots, r_m^T$) is a subspace since it is a span.

$\vec{x} \in$ Null(A) means

$$\vec{0} = A\vec{x} = \begin{bmatrix} \vec{r}_1 \cdot \vec{x} \\ \vec{r}_2 \cdot \vec{x} \\ \vdots \\ \vec{r}_m \cdot \vec{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\vec{x} \perp \vec{r}_i$ all Rows $\vec{r}_i$



So $\text{NULL}(A)$ = all vectors orthogonal to Row space.

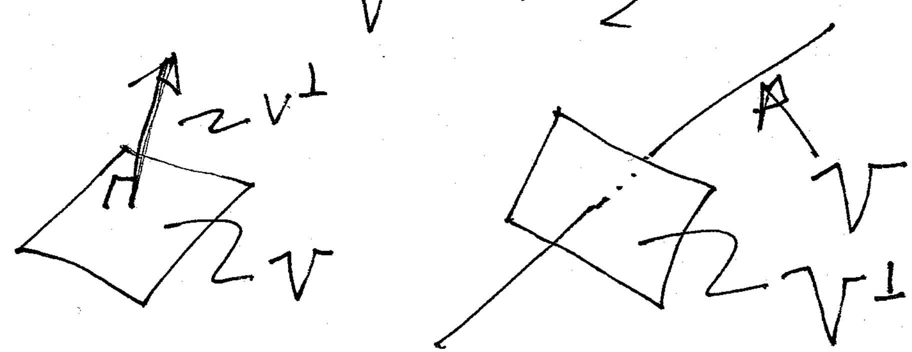
Since $\vec{x} \perp \vec{r}_i$ for all i

$$\vec{x} \cdot \underbrace{\left(\sum \alpha_i \vec{r}_i\right)}_{\text{Row space}} = \sum \alpha_i (\vec{x} \cdot \vec{r}_i) = \sum \alpha_i 0 = 0$$

DEF If V is a subspace,

$$V^{\perp} = \{\vec{x} : \vec{x} \cdot \vec{v} = 0, \text{ for all } \vec{v} \in V\}$$

orthogonal complement



HW: If V is a subspace, so is $V^{\perp}$.

$\Rightarrow$ $\text{Null}(A) = \text{Row}(A)^{\perp}$

Subspace $\Leftarrow$ Subspace

This way of thinking of (matrix) x (vector) thinks of A as being made up of **Rows**.

~~Next~~ way thinks of A as being made up of columns.

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 2 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

$$= 1 \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \qquad \vec{c}_i \in \mathbb{R}^m$$

$$\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1 \vec{c}_1 + x_2 \vec{c}_2 + \cdots + x_n \vec{c}_n$$

Linear combin of col. of A

The next important subspace is

$$\mathrm{col}(A) = \mathrm{span}(\vec{c}_1, \ldots, \vec{c}_n)$$

$=$ linear combin of col. of A.

So considering all input vector vectors, $A\vec{x}$ = all output vectors

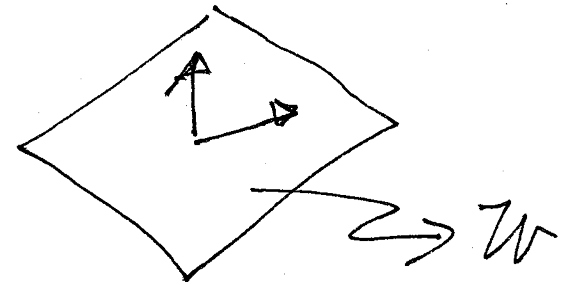
= all lin com of col of $A = \mathrm{col}(A)$

also called the range(A)

Subspace basis

$\mathcal{C} = \{\vec{w}_1, \dots, \vec{w}_k\}$ is a basis for the <u>subspace</u> W if <u>every $\vec{w} \in W$</u> can be written uniquely as

$$\vec{w} = \sum_{i=1}^{k} \alpha_i \vec{w}_i$$

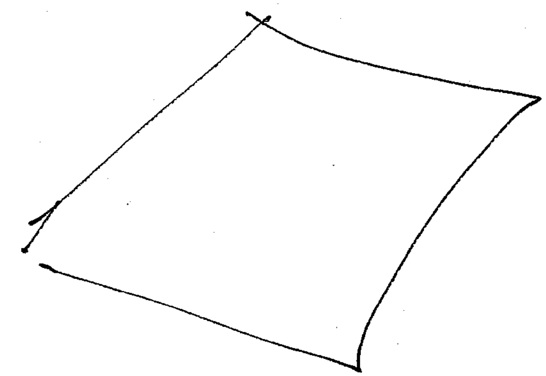


This happens exactly when

(1) $\text{span}(\vec{w}_1, \dots, \vec{w}_k) = W$

(2) $\vec{w}_i$ are lin. ind.

FACT: Every basis of W has the same number of elements — This number is the dimension of W.



dim = # of independent features in the data.

DEF A is matrix

$\mathrm{rank}(A) = \dim(\mathrm{col}(A))$

Theorem $\dim(\mathrm{row}(A)) = \mathrm{rank}(A)$.