Review of complex numbers and functions

\[ e^{i\theta} = \cos \theta + i \sin \theta \]
\[ e^z = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y) \]

Polar representation of complex numbers:

\[ z = re^{i\theta}, \quad r = |z|, \quad \tan \theta = \frac{y}{x} \]

\[ r e^{i\theta} = r (\cos \theta + i \sin \theta) \]

\[ e^{i\theta} : 0 \leq \theta < 2\pi \] is the unit circle \[ |z| = 1 \]

\[ e^{i\theta} = \cos \theta + i \sin \theta \Rightarrow \cos \theta = \frac{(e^{i\theta} + e^{-i\theta})}{2} \]

\[ \sin \theta = \frac{(e^{i\theta} - e^{-i\theta})}{2i} \]
For our next topic of Fourier methods and Convolutional nets a broader view of Linear Algebra is needed.

- On what structures can we do Linear Algebra?
  - $V$ is a vector space means there are two operations, a way of adding vectors and a way of rescaling vectors. The rescaling factors come from the scalar field $F$, which for us will always be $\mathbb{R}$ (real numbers) and $\mathbb{C}$ (complex numbers).

- The two operations are required to satisfy a long list of properties you learned in your 1st Linear Algebra course.
The most important are

\[ \mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \quad \text{(commutative)} \]
\[ (\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w}) \quad \text{(associative)} \]
\[ \alpha (\mathbf{u} + \mathbf{v}) = \alpha \mathbf{u} + \alpha \mathbf{v} \quad \text{(distributive)} \]

Examples - Thus far we have been in \( \mathbb{R}^n \).

= collection of all \( n \)-tuples as column vectors

\[ \mathbf{u} + \mathbf{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{pmatrix} \]

The scalar field is \( \mathbb{R} \)

\[ \alpha \mathbf{u} = \alpha \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix} \]
Discrete Fourier Analysis takes place in $\mathbb{C}^n = \text{all n-tuples of complex numbers with } F = \mathbb{C}$, the scalar field.

Sane for most as real case.

\[ \begin{pmatrix} 1+i \end{pmatrix} + \begin{pmatrix} 2-1i \end{pmatrix} = \begin{pmatrix} 3 \\ 1+4i \end{pmatrix} \]

\[ (1-i) \begin{pmatrix} 1+i \end{pmatrix} = i+1 = \begin{pmatrix} 2 \\ 1+i \end{pmatrix} \]

- Fourier Series use $L^2([\pi, \pi])$ which is all complex valued functions $f: [\pi, \pi] \to \mathbb{C}$ with $\int_{-\pi}^{\pi} |f(x)|^2 dx < \infty$ and $F = \mathbb{C}$, $(\alpha f)(x) = \alpha (f(x))$. 
- In any vector space, linear combinations and basis work the same (almost! A basis for \( L^2[-\pi, \pi] \) contains infinitely many elements, so a linear combination involves a limit - more on this later)

- Linear transformations are the same \( L : V \to W \)

A linear transformation is

\[ L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2) \]

- In \( \mathbb{R}^n \) and \( \mathbb{C}^n \), linear transformations are represented by matrices.

For \( \mathbb{C}^n \) the matrices have complex entries

\[ M = \begin{bmatrix} 2 + i & 3i & 7 - 2i \\ 2 - i & 5 & 8 + 7i \end{bmatrix} \]
Inner products work more or less the same but for the complex case things are a bit different.

\[ u, v \in \mathbb{C}^n, \quad \langle u, v \rangle = \overline{u_1}v_1 + \ldots + \overline{u_n}v_n \quad \text{where} \]

\[ \overline{2} = x - iy \quad \text{when} \quad 2 = x + iy \quad \text{so inner product is complex valued} \]

\[ \langle \begin{bmatrix} 2i \\ 3+i \end{bmatrix}, \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \rangle = \overline{2i} \cdot (1+i) + (3+i)2 = 2i \cdot 1 + 2 + 6 - 2i = 8 - 4i \]

Notice \[ 2 \overline{2} = (x+iy)(x-iy) = x^2 + y^2 = |2|^2 \]

So \[ \langle u, v \rangle = \overline{u_1}v_1 + \ldots + \overline{u_n}v_n = |u_1|^2 + \ldots + |u_n|^2 \]

As it should, which partially explains the form of \( \langle , \rangle \).
Notice two new features

\[ \langle u, v \rangle = \overline{\langle v, u \rangle} \]

and \[ \langle u, v \rangle = \overline{\alpha} \langle u, v \rangle \]

while \[ \langle u, \alpha v \rangle = \alpha \langle u, v \rangle \]

Because of these differences \( \langle , \rangle \) is called a **Hermitian Inner Product**

The space \( L^2[-\pi, \pi] \) has a Hermitian Inner Product

\[ \langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(x)} g(x) \, dx \]

Inner products give a norm

\[ \| v \| = \langle v, v \rangle^{1/2} \]

So in \( L^2[-\pi, \pi] \)

\[ \| f \|_2 = \left( \int_{-\pi}^{\pi} \overline{f(x)} f(x) \, dx \right)^{1/2} \]
Examples

$\langle e^{it}, e^{it} \rangle = \int_{-\pi}^{\pi} e^{it} \overline{e^{it}} \, dt = \int_{-\pi}^{\pi} 1 \, dt = 2\pi$

$|e^{it}| = \sqrt{\int_{-\pi}^{\pi} |e^{it}|^2 \, dt} = \sqrt{2\pi}$

$e^{it}$ and $e^{it}$ are orthogonal because

$\int_{-\pi}^{\pi} e^{it} \overline{e^{it}} \, dt = \int_{-\pi}^{\pi} 1 \, dt = 2\pi$

So

$\|e^{it}\|^2 = \int_{-\pi}^{\pi} |e^{it}|^2 \, dt = 2\pi$

$\|e^{it}\| = \sqrt{2\pi}$
Complex matrices

Instead of the transpose as used in real matrices, complex matrices use the conjugate transpose or adjoint written $A^*$ or $A^\dagger$.

$$A^* = A^T \quad \text{not} \quad (A^*)^* = A$$

$$A = \begin{bmatrix} 2i & 1+i \\ 7-3i & 2 \end{bmatrix} \quad A^* = \begin{bmatrix} -2i & 7+3i \\ 1-i & 2 \end{bmatrix}$$

Why the conjugate? It is because of the form of the Hermitian inner product.
\[ \langle \hat{u}, \hat{v} \rangle = \sum_{i=1}^{n} \hat{u}_i \hat{v}_i \]. If we treat, as usual, \( \hat{u} \) and \( \hat{v} \) as column vectors we can also write

\[ \langle \hat{u}, \hat{v} \rangle = \overrightarrow{\hat{u}}^T \overleftarrow{\hat{v}} \]  
conjugate transpose

Now let's get a matrix involved

\[ \langle A \hat{u}, \hat{v} \rangle = (A \hat{u})^T \hat{v} = \overrightarrow{\hat{u}}^T \overrightarrow{A^T} \hat{v} \]

\[ = \overrightarrow{\hat{u}}^T (A^T \hat{v}) = \overrightarrow{\hat{u}}^T A^* \hat{v} = \langle \hat{u}, A^* \hat{v} \rangle \]

So when you pull a matrix across an inner product to the "adjoint" position the matrix transforms to its adjoint.

It works the other way also so

\[ \langle \hat{u}, A \hat{v} \rangle = \langle A^* \hat{u}, \hat{v} \rangle \]
The analog of an orthogonal matrix is called **unitary** if \( U^* = U^{-1} \).

The analog of a symmetric matrix is called **Hermitian** if \( U^* = U \) (or self-adjoint).

With the appropriate changes, unitary and Hermitian matrices have properties like orthogonal and symmetric ones.

- **Theorem:** Unitary matrices preserve the Hermitian inner product.

**Proof:** \( \langle U \hat{x}, U \hat{y} \rangle = \langle U^* U \hat{x}, \hat{y} \rangle \)

\[ = \langle U^* U \hat{x}, \hat{y} \rangle = \langle \hat{x}, \hat{y} \rangle = \langle \hat{x}, \hat{y} \rangle \]

- **Theorem:** \( U \) is unitary \( \iff \) its columns form an orthonormal basis w.r.t. the Hermitian inner product.
The Spectral Theorem: If $A$ is Hermitian $\Rightarrow$

1. All its eigenvalues are real numbers.
2. There is a unitary matrix $U$ so that $A = U \text{diag}(\lambda_1, \ldots, \lambda_n) U^*$ or the eigenvectors form an orthonormal basis.

Proof: (1) Say $\lambda$ is an eigenvalue with eigenvector $x \neq 0$.

\[ \langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle = \lambda \|x\|^2 \]

But also $\langle x, Ax \rangle = \langle A^*x, x \rangle = \langle A_1x, x \rangle = \langle x, x \rangle$

\[ = \overline{\lambda} \|x\|^2 = \lambda \|x\|^2 \text{ since } \|x\|^2 \neq 0 \]

$\lambda = \overline{\lambda}$ so $x$ is real.

(2) is similar to the real case.