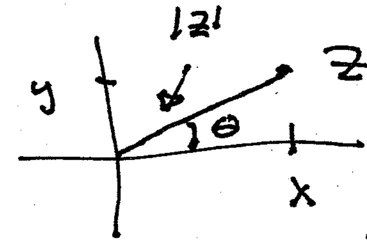
Review of complex numbers and functions

C1

$z = x + iy$



$z = x+iy = |z|e^{i\theta}$

$\bar{z} = x - iy$

$\bar{z}$ or z^* is the conjugate $\bar{z} = x - iy$, $\bar{\bar{z}} = z$

norm or modulus $|z| = \sqrt{x^2+y^2} = \sqrt{z\bar{z}}$, $|zw| = |z||w|$

z is real $\Leftrightarrow$ $z = \bar{z}$

$e^{i\theta} = \cos\theta + i\sin\theta$ (Euler's formula)

$\overline{e^{i\theta}} = \cos\theta - i\sin\theta = e^{-i\theta}$

$|e^{i\theta}| = 1 = \sqrt{\cos^2\theta + \sin^2\theta} = \sqrt{1}$

$e^{i(\theta+2\pi)} = e^{i\theta}$, $e^{i\pi} = -1$

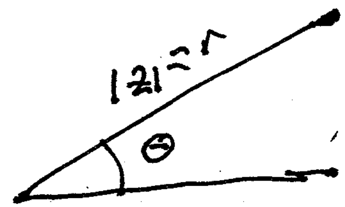
C2

$$e^{z} = e^{x+iy} = e^{x} e^{iy} = e^{x}(\cos y + i \sin y)$$

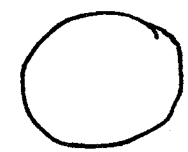
polar representation of complex numbers

$$z = r e^{i\theta}, \quad r = |z|, \quad \tan\theta = y/x$$

$$re^{i\theta} = r(\cos\theta + i\sin\theta)$$



$\{ e^{i\theta} : 0 \le \theta \le 2\pi \}$ is the unit circle $|z| = 1$



$$e^{i\theta} = \cos\theta + i\sin\theta \implies \cos\theta = (e^{i\theta} + e^{-i\theta})/2$$

$$\sin\theta = (e^{i\theta} - e^{-i\theta})/2i$$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

- For our next topic of Fourier methods and Convolutional Nets a broader view of Linear Algebra is needed
- On what structures can we do Linear Algebra
- V is a vector space means there are two operations, a way of adding vectors and a way of rescaling vectors. The rescaling factors come from the <u>scalar field</u> F, which for us will always be $\mathbb{R}$ (real numbers) and $\mathbb{C}$ (complex numbers).
- The two operations are required to satisfy a long list of properties you learned in your 1st Linear Algebra course.

VS

- The most important are

$$\vec{u} + \vec{v} = \vec{v} + \vec{u} \quad \text{(commutative)}$$
$$(\vec{u}+\vec{v})+\vec{w} = \vec{u} + (\vec{v}+\vec{w}) \quad \text{(associative)}$$
$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{u} + \alpha\vec{v} \quad \text{(distributive)}$$

- Examples – Thus far we have been in $\mathbb{R}^n$
  = collection of all n-tuples as column vectors

$$\vec{u}+\vec{v} = \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} + \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} u_1+v_1 \\ \vdots \\ u_n+v_n \end{pmatrix}$$

The scalar field is $\mathbb{R}$

$$\alpha\vec{u} = \alpha \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = \begin{pmatrix} \alpha u_1 \\ \vdots \\ \alpha u_n \end{pmatrix}$$

- Discrete Fourier Analysis takes place in $\mathbb{C}^n$ = all n-tuples of complex numbers with $F = \mathbb{C}$, the scalar field.

Same formulas as real case

eg $\begin{pmatrix} 1+i \\ i \end{pmatrix} + \begin{pmatrix} 2-i \\ 1+3i \end{pmatrix} = \begin{pmatrix} 3 \\ 1+4i \end{pmatrix}$

$(1-i)\begin{pmatrix} 1+i \\ i \end{pmatrix} = \begin{matrix} 1+1 \\ i+1 \end{matrix} = \begin{pmatrix} 2 \\ 1+i \end{pmatrix}$

- Fourier Series use $L^2([-\pi,\pi])$ which is all complex valued functions $f:[-\pi,\pi]\to\mathbb{C}$ with $\int_{-\pi}^{\pi}|f(x)|^2dx < \infty$ and $F = \mathbb{C}$, $(\alpha f)(x) = \alpha\, f(x)$

- In any vector space, linear combinations and basis work the same (almost! a basis for $L^2[-\pi, \pi]$ contains infinitely many elements, so a linear combination involves a limit - more on this later)
- Linear transformations are the same $L : V \to W$ is a linear transformation If
$$L(\alpha v_1 + \beta v_2) = \alpha L(v_1) + \beta L(v_2)$$
- In $\mathbb{R}^n$ and $\mathbb{C}^n$, Linear transformations are represented by matrices, for $\mathbb{C}^n$ the matrices have complex entries

$$M = \begin{bmatrix} 2+i & 3i & 7-2i \\ 2-i & 5 & 8+7i \end{bmatrix}$$

- Inner products work more or less the same but for the complex case things are a bit different

- $u, v \in \mathbb{C}^n$, then $\langle u,v\rangle = \bar{u}_1 v_1 + \dots + \bar{u}_n v_n$ where $\bar{z} = x - iy$ when $z = x + iy$ so inner product is complex valued

$$\left\langle \begin{bmatrix} 2i \\ 3+i \end{bmatrix}, \begin{bmatrix} 1+i \\ 2 \end{bmatrix} \right\rangle = \overline{2i}\,(1+i) + \overline{(3+i)}\,2$$

$$= -2i(1+i) + (3-i)2 = -2i + 2 + 6 - 2i = 8 - 4i$$

Notice $z\bar{z} = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

So $\langle u,u\rangle = \bar{u}_1 u_1 + \dots + \bar{u}_n u_n = |u_1|^2 + \dots + |u_n|^2$

as it should, which partially explains the form of $\langle\, ,\, \rangle$.

Notice two new features

$$\langle u, v \rangle = \overline{\langle v, u \rangle}$$

and $\langle \alpha u, v \rangle = \bar{\alpha} \langle u, v \rangle$

while $\langle u, \alpha v \rangle = \alpha \langle u, v \rangle$

Because of these differences $\langle\, , \rangle$ is called a <u>Hermitian inner product</u>

- The space $L^2[-\pi, \pi]$ has a Hermitian Inner Produ

$$\langle f, g \rangle = \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$$

- Inner products give a norm $\|\vec{v}\| = \langle v, v \rangle^{1/2}$

So in $L^2(-\pi, \pi)$ $\|f\|_2 = \left(\int_{-\pi}^{\pi} \bar{f}(x) f(x)\, dx \right)^{1/2}$

$$= \left(\int_{-\pi}^{\pi} |f(x)|^2 dx \right)^{1/2}$$

Examples

$$\langle e^{it}, e^{2it} \rangle = \int_{-\pi}^{\pi} \overline{e^{it}}\, e^{2it}\, dt$$

$$= \int_{-\pi}^{\pi} e^{-it} e^{2it}\, dt = \int_{-\pi}^{\pi} e^{it}\, dt$$

$$= \frac{e^{it}}{i}\Bigg|_{-\pi}^{\pi} = \frac{e^{i\pi} - e^{-i\pi}}{i} = \frac{-1 - -1}{i} = 0$$

So they are orthogonal

$$\| e^{it} \| = \left(\int_{-\pi}^{\pi} \overline{e^{it}}\, e^{it}\, dt \right)^{1/2} = \left(\int_{-\pi}^{\pi} e^{-it} e^{it}\, dt \right)^{1/2}$$

$$= \left(\int_{-\pi}^{\pi} e^{0}\, dt \right)^{1/2} = \left(\int_{-\pi}^{\pi} dt \right)^{1/2} = \sqrt{2\pi}$$

Complex matrices

Instead of the transpose as used in real matrices complex matrices use the conjugate transpose or adjoint written A^* or A^H

$$A^* = \overline{A^T} \qquad \text{notice } (A^*)^* = A$$

$$A = \begin{bmatrix} 2i & 1+i \\ 7-3i & 2 \end{bmatrix} \quad A^* = \begin{bmatrix} -2i & 7+3i \\ 1-i & 2 \end{bmatrix}$$

why the conjugate? It is because of the form of the ~~inner~~ Hermitian inner product

$\langle \vec{u}, \vec{v} \rangle = \sum_{i=1}^{n} \overline{u_i}\, v_i$. If we treat, as usual, $\vec{u}$ and $\vec{v}$ as column vectors we can also write

$$\langle \vec{u}, \vec{v} \rangle = \overline{\vec{u}}^{T} \vec{v}$$

conjugate transpose

Now let's get a matrix involved

$$\langle A\vec{u}, \vec{v} \rangle = \overline{(A\vec{u})}^{T} \vec{v} = \overline{u}^{T} \overline{A}^{T} \vec{v}$$

$$= \overline{u}^{T} (\overline{A}^{T} \vec{v}) = \overline{u}^{T} A^{*} \vec{v} = \langle \vec{u}, A^{*} \vec{v} \rangle$$

so when you pull a matrix across an inner product to the "adjoint" position the matrix transforms to its adjoint

it works the other way also so

$$\langle \vec{u}, A\vec{v} \rangle = \langle A^{*}\vec{u}, \vec{v} \rangle$$

The analog of an orthogonal matrix is called <u>unitary</u> if $U^* = U^{-1}$

The analog of a symmetric matrix is called Hermitian (or self-adjoint) if $U^* = U$

with the appropriate changes, unitary and Hermitian matrices have properties like orthogonal and symmetric ones.

- <u>Theorem</u>: Unitary Matrices preserve the Hermitian Inner product

Proof: $\langle U\vec{x}, U\vec{y} \rangle = \langle U^* U \vec{x}, \vec{y} \rangle$

$= \langle U^{-1} U \vec{x}, \vec{y} \rangle = \langle I\vec{x}, \vec{y} \rangle = \langle \vec{x}, \vec{y} \rangle$

- <u>Theorem</u>: U is unitary $\Leftrightarrow$ its columns form an orthonormal basis w.r.t. the Hermitian inner product

The Spectral Theorem If A is Hermitian $\Rightarrow$

(1) All it's eigen values are real numbers

(2) There is a unitary matrix U so that

$$A = U \operatorname{diag}(\lambda_1, \ldots, \lambda_n) U^*$$

or the eigenvectors form an orthonormal basis

Proof (1) Say λ is an eigen value with eigen vector $\vec{x} \neq 0$.

$$\langle x, Ax \rangle = \langle x, \lambda x \rangle = \lambda \langle x, x \rangle = \lambda \|x\|^2$$

But also $\langle x, Ax \rangle = \langle A^* x, x \rangle \underset{\substack{A \text{ is} \\ \text{Hermitian}}}{=} \langle Ax, x \rangle = \langle \lambda x, x \rangle$

$$= \bar{\lambda} \langle x, x \rangle = \bar{\lambda} \|x\|^2$$

since $\|x\|^2 \neq 0$,

$\lambda = \bar{\lambda}$ so λ is Real.

(2) is similar to the real case.