A big idea is Data Science, DSP, image processing...

- **f** is a data vector, signal/image function
- **f** is in V a vector space.
- Find an orthonormal basis $\hat{z}_1, \hat{z}_2, \ldots$ for $V$ that encodes some properties of interest.
- Expand $\hat{f}$ in terms of the basis
  $$\hat{f} = d_1 \hat{z}_1 + d_2 \hat{z}_2 + \ldots$$
  Note

1. $d_j$ is the amount of $\hat{f}$ in $\hat{z}_j$
2. Truncating the expansion
   $$\hat{f}(x) = d_1 \hat{z}_1 + \ldots + d_k \hat{z}_k$$
   stores an efficient, lower dim version of $\hat{f}$ which still encodes essential information
Where do we get the orthonormal basis:

- Eigen vectors of Hermitian matrix (operator)
- Gram-Schmidt process on another basis
  - \( \text{Science, \ldots} \)

Example 1: Let \( V \) be all polynomials with real coefficients defined on \([-1, 1]\). Then

\[
1, t, t^2, \ldots \quad 3 \text{ is a basis}
\]

Since any polynomial can be written

\[
p(t) = q_0 \cdot 1 + q_1 \cdot t + \ldots + q_n \cdot t^n
\]

(That is the definition of a polynomial)

Now put the inner product on \( V \)

\[
\langle p, z \rangle = \sum_{-1}^{1} p(t)z(t) dt
\]
Then using Gram-Schmidt on the given basis \( \{3, 4, 1\} \)
yields an orthonormal basis for \( V \):

\[ \psi_0(x) = \sqrt{\frac{1}{2}}, \quad \psi_1(x) = \sqrt{\frac{3}{2}} x, \quad \psi_2(x) = \frac{1}{2} \sqrt{\frac{5}{2}} (3x^2 - 1), \quad \ldots \]

These are called the Legendre polynomials.

**Example 2:** Let \( V = L^2([-\pi, \pi]) \) with the Hermitian inner product

\[ \langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) \, dx \]

The Fourier basis is orthonormal

\[ \frac{e^{-inx}}{\sqrt{2\pi}}, \frac{e^{-3ix}}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{e^{ix}}{\sqrt{2\pi}}, \frac{e^{3ix}}{\sqrt{2\pi}}, \ldots, \frac{e^{inx}}{\sqrt{2\pi}} \]

Note that these are indexed by all integers, not just positive ones and recall

\[ e^{inx} = \cos nx + i \sin nx \]
Let's check they are orthonormal:

\[ \langle e_{i(n-m)}^n, e_{i(n-m)}^m \rangle = \frac{1}{2\pi} \int_0^{2\pi} e^{i(n-m)x} \overline{e^{i(n-m)x}} \, dx = \frac{1}{2\pi} \int_0^{2\pi} 1 \, dx = \pi \delta_{n-m} - 1 \]

This shows o.k. Shown it is a basis is harder.
Example 3: The discrete Fourier basis is an orthonormal basis for \( \mathbb{C}^n \) obtained by discretizing the usual Fourier basis. It is the subject of the next few lectures.

Recall that we want to use the o.u. basis to give coordinates to vectors. The next theorem says how to do that.

Theorem: If \( \vec{z}_1, \vec{z}_2, \ldots, \vec{z}_n \) is an o.u. basis w.r.t. a Hermitian inner product \( \langle \cdot, \cdot \rangle \), then

\[
\vec{v} = \sum_{i=1}^{n} |v_i| \vec{z}_i
\]

with each \( \alpha_{|z_i|} = \langle \vec{z}_i, \vec{v} \rangle \).
We have since $E_2, \ldots, 3$ is a basis that
for some $d_1, d_2, \ldots$, we have $V = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle + \ldots$
then using the linearity of the Hermitian inner product:

$$\langle 2_{k1}, \tilde{v} \rangle = \langle 2_{k1}, x_{k1} \rangle + \langle 2_{k1}, x_{k2} \rangle + \ldots$$
$$= d_1 \langle 2_{k1}, x_{k1} \rangle + d_2 \langle 2_{k1}, x_{k2} \rangle + \ldots$$
$$= 0 + 0 + \ldots + 0 = 0$$

Since $E_2, \ldots, 3$ is an orthonormal basis.
Example: For the Fourier basis for $L^2[-\pi, \pi]$

If $f(t) = \sum \alpha_n e^{int}$

$\alpha_n = \langle e^{-int}, f(t) \rangle = \int_{-\pi}^{\pi} e^{-int} f(t) dt$

$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t) e^{-int} dt$ (usual to write it in this order).

Now let $f(t) = \begin{cases} 1 & \text{when } |t| \leq \frac{\pi}{2} \\ 0 & \text{when } \frac{\pi}{2} < |t| \leq \pi \end{cases}$

We want to express $f$ in the Fourier basis, i.e. find its Fourier expansion.
\[ \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} f(t) e^{-int} dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{int} dt \]

\[ = \frac{-1}{\sqrt{2\pi} in} \left( e^{-i\pi/2} - e^{i\pi/2} \right) = -\frac{2i\sin(n\pi/2)}{\sqrt{2\pi} in} \]

Using Euler's formula.

Notice this is 1, so \( n \neq 0 \), do \( n = 0 \) separately.

\[ \alpha_0 = \frac{1}{\sqrt{2\pi}} \int_{-\pi/2}^{\pi/2} e^{0it} dt = \frac{\pi}{\sqrt{2\pi}} \]
\[ f(\pi) = \left( \frac{1}{\sqrt{2\pi}} \right) \frac{1}{\sqrt{2\pi}} + \sum_{n \neq 0} \left( 2 \frac{\sin(n\pi/2)}{n\pi} \right) e^{i n \pi} \]

\[ = \frac{1}{2} + \sum_{n \neq 0} \left( \frac{\sin(n\pi/2)}{n\pi} \right) e^{i n \pi} \]

Where convergence is a deeper issue. Now notice that the non-zero terms come in \( n + n, -n \) pairs and
\[ \sin(-n\pi/2) = -\frac{\sin(n\pi/2)}{n\pi} \]

so adding the pairs
\[ a_n e^{-in\pi} + a_n e^{in\pi} = 2 \frac{\sin(n\pi/2)}{n\pi} \left( \cos(-n\pi) + i \sin(-n\pi) \right) \]

\[ + \frac{\sin(n\pi/2)}{n\pi} \left( \cos(-n\pi) + i \sin(n\pi) \right) \]

\[ = 2 \frac{\sin(n\pi/2)}{n\pi} \cos(n\pi) \quad \text{Thus we have expansion} \]

\[ f(\pi) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2\sin(n\pi/2)}{n\pi} \cos(n\pi) = \frac{1}{2} + \frac{2}{\pi} \cos \frac{\theta}{2} - \frac{2}{3\pi} \cos 3\frac{\theta}{2} + \ldots \]

\[ \forall \theta \in \mathbb{R} \]

\[ MOVIE \]
There are two more fundamental facts about inner products.

The Pythagorean Theorem: If \( \mathbf{v}_1, \ldots, \mathbf{v}_n \) is an orthonormal set, then if \( \mathbf{v} \) is any vector with respect to the inner product, then if \( \mathbf{v} = \sum \mathbf{v}_i \), we have as usual, \( \| \mathbf{v} \|^2 = \sum \| \mathbf{v}_i \|^2 \).

Proof: \[
\langle \mathbf{v}, \mathbf{v} \rangle = \langle \sum \mathbf{v}_i, \sum \mathbf{v}_i \rangle = \sum \langle \mathbf{v}_i, \mathbf{v}_i \rangle \]

and \( \langle \mathbf{v}, \mathbf{v} \rangle = 0 \) if \( \mathbf{v} = 0 \).

Hence, \( \sum \| \mathbf{v}_i \|^2 = \| \mathbf{v} \|^2 \).
In signal processing and compression you often want a representation of the signal that only uses the lower, more significant harmonics or frequencies. What should this be?

The next theorem says that the least squares best fit comes from truncating the O.N. expansion.

So if \( f(x) = \sum_{n=0}^{\infty} c_n \frac{\sin(n \pi x)}{n \pi} \)

The best order \( N \) fitting trig polynomial is:

\[
\sum_{n=-N}^{N} c_n \frac{\sin(n \pi x)}{n \pi}
\]
Theorem: Say $\{\vec{z}_1, \ldots, \vec{z}_M\}$ is an orthonormal set

$$\vec{v} = \sum_{i=1}^{M} \alpha_i \vec{z}_i$$

Now fix an order $N$

and let $\vec{v}(N) = \sum_{i=1}^{N} \alpha_i \vec{z}_i$.

Then for any other order $N$ expansion

$$\vec{w} = \sum_{i=1}^{N} \beta_i \vec{z}_i$$

$$\|\vec{v} - \vec{v}(N)\| \leq \|\vec{v} - \vec{w}\|$$

with equality only when $\vec{w} = \vec{v}(N)$.

Thus $\vec{v}(N)$ gives the best order $N$ approximation.
Proof we compute both sides using the Pythagorean Theorem

\[
\| \vec{v} - \vec{w} \| = \| \sum_{i=1}^{M} \vec{a}_i \vec{z}_i - \sum_{i=1}^{N} \vec{b}_i \vec{z}_i \| = \| \sum_{i=1}^{M} \vec{a}_i \vec{z}_i \| \]

\[
= \sum_{i=1}^{M} |a_i|^2 \quad (1)
\]

\[
\| \vec{v} - \vec{w} \| = \| \sum_{i=1}^{N} \beta_i \vec{z}_i \| = \| \sum_{i=1}^{M} \beta_i \vec{z}_i \| = \| \sum_{i=1}^{M} \vec{a}_i \vec{z}_i \| + \| \sum_{i=M+1}^{N} \beta_i \vec{z}_i \| \]

\[
= \sum_{i=1}^{M} |a_i|^2 + \sum_{i=M+1}^{N} |\beta_i|^2 \quad (2)
\]

So \[
\| \vec{v} - \vec{w} \|^2 - \| \vec{v} - \vec{w} \| = \sum_{i=1}^{M} |a_i|^2 - |a_i|^2 \geq 0
\]

and is equal to zero only when \( a_i = \beta_i \) for \( i = 1, \ldots, M \)

or when \( \vec{w} = \vec{v} / |v| \).