A big idea is Data Science, DSP, image processing, scientific computation ....

$\vec{f}$ is a data vector, signal, image, function

$\vec{f}$ is in V a vector space

- Find an orthonormal basis $\vec{q}_1, \vec{q}_2, \ldots$ for V that encodes some properties of interest.
- Expand $\vec{f}$ in terms of the basis

$$\vec{f} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \cdots$$

Then

(1) α_j is the amount of $\vec{f}$ in $\vec{q}_j$

(2) Truncating the expansion
$\vec{f}(k) = \alpha_1 \vec{q}_1 + \cdots + \alpha_k q_k$ stores an efficient, lower dim version of $\vec{f}$ which still encodes essential information

Where do we get the orthonormal basis

- Eigenvectors of Hermitian matrix (operator)
- Gram-Schmidt process on another basis
- Science, ...

Example 1: Let V be all polynomials with real coefficents defined on $[-1,1]$. Then

$$\{1, t, t^2, \dots\} \text{ is a basis}$$

Since any polynomial can be written

$$p(t) = a_0 \cdot 1 + a_1 \cdot t + \dots + a_n t^n$$

(That is the definition of a polynomial)

Now put the inner product on V

$$\langle p, q \rangle = \int_{-1}^{1} p(t) q(t)\, dt$$

Then using Gram-Schmidt on the given basis $\boxed{3t}$ 4
yields an <u>ortho normal basis</u> for V

$$\varphi_0(t) = \sqrt{\tfrac{1}{2}}, \quad \varphi_1(t) = \sqrt{\tfrac{3}{2}}\, t, \quad \varphi_2(t) = \frac{1}{2}\sqrt{\frac{5}{2}}\,(3t^2-1), \ldots$$

These are called the <u>Legendre</u> polynomials.

<u>Example 2</u>: Let $V = L^2([-\pi, \pi])$ with the Hermitian inner product $\langle f, g\rangle = \int_{-\pi}^{\pi} \overline{f(x)}\, g(x)\, dx$

The Fourier basis is orthonormal

$$\frac{e^{-nxi}}{\sqrt{2\pi}} \cdots, \frac{e^{-2xi}}{\sqrt{2\pi}}, \frac{e^{-xi}}{\sqrt{2\pi}}, \frac{1}{\sqrt{2\pi}}, \frac{e^{xi}}{\sqrt{2\pi}}, \frac{e^{2xi}}{\sqrt{2\pi}}, \frac{e^{3xi}}{\sqrt{2\pi}}, \cdots \frac{e^{nxi}}{\sqrt{2\pi}}$$

NOTE that these are indexed by all integers not just positive ones and recall

$$e^{inx} = \cos nx + i \sin nx.$$

Let's check they are orthonormal

If $m \neq n$

$$\left\langle \frac{e^{imx}}{\sqrt{2\pi}}, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} \overline{e^{imx}}\, e^{inx}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(n-m)x}\, dx = \frac{1}{2\pi} \left. \frac{e^{i(n-m)x}}{i(n-m)} \right|_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi i(n-m)} \Big[e^{i(n-m)\pi} - e^{-i(n-m)\pi} \Big] = 0$$

$-1 \quad - -1$ when $(n-m)$ is odd

$1 \quad - \quad 1$ when $(n-m)$ is even

and $$\left\langle \frac{e^{inx}}{\sqrt{2\pi}}, \frac{e^{inx}}{\sqrt{2\pi}} \right\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-inx} e^{inx}\, dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} 1\, dx = \frac{2\pi}{2\pi} = 1$$

This shows o.n., showing it is a basis is harder.

<u>Example 3</u>: The discrete Fourier basis is an orthonormal basis for $\mathbb{C}^n$ obtained by discretizing the usual Fourier basis. It is the subject of the next few lectures

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Recall that we want to use the o.n. basis to give coordinates to vectors. The next theorem says how to do that

<u>Theorem</u>: If $\{\vec{q}_1, \vec{q}_2, \dots\}$ is an o.n. basis w.r.t. a Hermitian inner product then

$$\vec{v} = \sum \alpha_i \vec{q}_i \quad \text{with each } \alpha_i = \langle \vec{q}_i, \vec{v} \rangle$$

Proof: We have since $\{q_1, \ldots\}$ is a basis that for some α_i,

$$\vec{v} = \alpha_1 \vec{q}_1 + \alpha_2 \vec{q}_2 + \cdots$$

then using the linearity of the Hermitian Inner Product

$$\langle q_k, \vec{v} \rangle = \langle \vec{q}_k, \alpha_1 \vec{q}_1 \rangle + \langle q_k, \alpha_2 \vec{q}_2 \rangle + \cdots + \langle \vec{q}_k, \alpha_k \vec{q}_k \rangle + \cdots$$

$$= \alpha_1 \langle \vec{q}_k, \vec{q}_1 \rangle + \alpha_2 \langle \vec{q}_k, \vec{q}_2 \rangle + \cdots + \alpha_k \langle \vec{q}_k, \vec{q}_k \rangle$$

$$= 0 + 0 + \cdots + \alpha_k$$

since $\{\vec{q}_1, \vec{q}_2, \ldots\}$ is an o.n. basis.

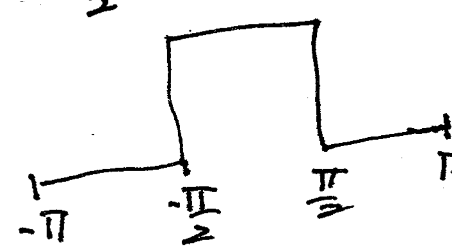
Example: For the Fourier basis for $L^2[-\pi, \pi]$

If $f(t) = \sum \alpha_n \frac{e^{int}}{\sqrt{2\pi}}$

$$\Rightarrow \quad \alpha_n = \left\langle \frac{e^{int}}{\sqrt{2\pi}}, f(t) \right\rangle = \int_{-\pi}^{\pi} \overline{\frac{e^{int}}{\sqrt{2\pi}}}\, f(t)\, dt$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$ (usual to write it in this order).

Now let $f(t) = \begin{cases} 1 & \text{when } |t| \leq \pi/2 \\ 0 & \text{when } \frac{\pi}{2} < |t| \leq \pi \end{cases}$



We want to express f is the Fourier basis, i.e. find its Fourier expansion.

$$\alpha_n = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} e^{-int}\, dt$$

$$= \frac{-1}{\sqrt{2\pi}\, in}\left(e^{-in\pi/2} - e^{in\pi/2} \right) = -\frac{-2i\sin(n\pi/2)}{\sqrt{2\pi}\, in} =$$

$$\frac{2\sin(n\pi/2)}{\sqrt{2\pi}\, n}$$ using Eulers formula

notice this n, so $n \neq 0$, do $n = 0$ separately

$$\alpha_0 = \frac{1}{\sqrt{2\pi}}\int_{-\pi/2}^{\pi/2} e^{0it}\, dt = \frac{\pi}{\sqrt{2\pi}}$$

So $f(t)$ "=" $\left(\frac{\pi}{\sqrt{2\pi}}\right)\cdot\frac{1}{\sqrt{2\pi}} + \sum_{n\neq 0}\left(\frac{2\sin n\pi/2}{\sqrt{2\pi}\,n}\right)\left(\frac{e^{int}}{\sqrt{2\pi}}\right)$

$$= \frac{1}{2} + \sum_{n\neq 0}\left(\frac{\sin(n\pi/2)}{n\pi}\right)e^{int}$$

where convergence is a deeper issue. Now notice that the nonzero terms come in $+n, -n$ pairs and

$$\frac{\sin(-n\pi/2)}{-n\pi} = \frac{\sin(n\pi/2)}{n\pi}$$

So adding the pairs

$$\alpha_{-n}e^{-int} + \alpha_n e^{int} = \frac{\sin(-n\pi/2)}{-n\pi}\left(\cos(-nt) + i\sin(-nt)\right) + \frac{\sin(n\pi/2)}{n\pi}\left(\cos(nt) + i\sin nt\right)$$

$$= \frac{2\sin(n\pi/2)}{n\pi}\cos nt.$$

Thus we have expansion

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2\sin n\pi/2}{n\pi}\cos nt = \frac{1}{2} + \frac{2}{\pi}\cos t - \frac{2}{3\pi}\cos 3t + \cdots$$

$\boxed{\text{MOVIE}}$

There are two more fundamental facts about o.n. basis

<u>The Pythagorean Theorem</u>: If $\{\vec{q}_1, \ldots\}$ is an o.n. set with respect to the inner product $\langle\, , \rangle$ and $\|\vec{v}\|^2 = \langle \vec{v}, \vec{v} \rangle$ as usual, then if $\vec{v} = \sum \alpha_i \vec{q}_i \Rightarrow \|\vec{v}\|^2 = \sum |\alpha_i|^2$,

<u>Proof</u>: $\langle \vec{v}, \vec{v} \rangle = \langle \sum \alpha_i \vec{q}_i , \sum \alpha_i \vec{q}_i \rangle$

$$= \sum_i \langle \alpha_i \vec{q}_i, \alpha_i \vec{q}_i \rangle + \sum_{i \neq j} \langle \alpha_i \vec{q}_i, \alpha_j \vec{q}_j \rangle$$

$$= \sum \bar{\alpha}_i \alpha_i \underbrace{\langle \vec{q}_i, \vec{q}_i \rangle}_{1} + \sum_{i \neq j} \bar{\alpha}_i \alpha_j \underbrace{\langle \vec{q}_i, \vec{q}_j \rangle}_{0}$$

$$= \sum |\alpha_i|^2$$

- In signal processing and compression you often want a representation of the signal that only uses the lower, more significant harmonics or frequencies. What should this be.

- The next theorem says that the least squares best fit comes from truncating the o.n. expansion.

- So if $f(t) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{int}}{\sqrt{2\pi}}$

The best order N fitting trig polynomial is

$$\sum_{n=-N}^{N} \alpha_n \frac{e^{int}}{\sqrt{2\pi}}$$

<u>Theorem</u>: Say $\{\vec{q}_1, \ldots\}$ is an orthonormal set

and $\vec{v} = \sum_{i=1}^{M} \alpha_i \vec{q}_i$. Now Fix an order N

and let $\vec{v}^{(N)} = \sum_{i=1}^{N} \alpha_i \vec{q}_i$ Then for any other

order N expansion $\vec{w} = \sum_{i=1}^{N} \beta_i \vec{q}_i$

$$\| \vec{v} - \vec{v}^{(N)} \| \leq \| \vec{v} - \vec{w} \|$$

with equality only when $\vec{w} = \vec{v}^{(N)}$.

Thus $\vec{v}^{(N)}$ gives the best order N approximation

Proof we compute both sides using the Pythagorean Theorem

$$\|\vec{v}-\vec{v}(N)\|^2 = \left\| \sum_{i=1}^{M} \alpha_i \vec{q}_i - \sum_{i=1}^{N} \alpha_i \vec{q}_i \right\|^2 = \left\| \sum_{i=N+1}^{M} \alpha_i \vec{q}_i \right\|^2$$

$$= \sum_{i=N+1}^{M} |\alpha_i|^2 \quad (1)$$

$$\|\vec{v}-\vec{w}\|^2 = \left\| \sum_{i=1}^{M} \alpha_i \vec{q}_i - \sum_{i=1}^{N} \beta_i \vec{q}_i \right\|^2 = \left\| \sum_{i=1}^{N} (\alpha_i-\beta_i)\vec{q}_i + \sum_{i=N+1}^{M} \alpha_i \vec{q}_i \right\|^2$$

$$= \sum_{i=1}^{N} |\alpha_i-\beta_i|^2 + \sum_{i=N+1}^{M} |\alpha_i|^2 \quad (2)$$

So $\|\vec{v}-\vec{w}\|^2 - \|\vec{v}-v(N)\|^2 = \sum_{i=1}^{N} |\alpha_i-\beta_i|^2 \geq 0$

and is equal to zero only when $\alpha_i=\beta_i$ for $i=1,\dots,N$

or when $\vec{w} = \vec{v}(N)$.