- The Discrete Fourier transform sends the data into discrete frequency space.

- If $x_0, \ldots, x_{N-1}$ is the data vector x

then $\hat{x} = DFT(x) = Fx$ where $F_{ij} = \frac{1}{\sqrt{N}} \omega_N^{-ij}$

- with $\omega_N = e^{2\pi i/N}$

- In formulas,

$$\hat{x}_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i kj/N}$$

- The $\hat{x}_j$ are usually complex numbers, so we plot $|\hat{x}_j|^2$ vs j

$\boxed{\text{Example}}$

Notice the symmetry about $\frac{N}{2}$ (assume now N is even)

FACT 1: If x is a real data vector, N is even

(a) $\hat{x}_{N-j} = \overline{\hat{x}_j}$, $j = 1, \ldots, \frac{N}{2} - 1$

(b) $\hat{x}_0 = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k$ and so is real

(c) $\hat{x}_{N/2} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k (-1)^k$ and so is real

Proof: Using the formula for the DFT

(a) $$\hat{x}_{N-j} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i k(N-j)/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i k} e^{2\pi i k j/N}$$

$$= \overline{\frac{1}{\sqrt{N}} \sum_{k=0} x_k e^{-2\pi i k j/N}} = \overline{\hat{x}_j}$$

Since x_k is real ($\bar{x}_k = x_k$) and $e^{-2\pi i k} = 1$

(b) $$\hat{x}_0 = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i k \cdot 0/N} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k$$

(c) $$\hat{x}_{\frac{N}{2}} = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k e^{-2\pi i k \frac{N}{2N}}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} x_k \left(e^{-\pi i}\right)^k = \frac{1}{\sqrt{N}} \sum_{k=0} (-1)^k x_k$$

- We now want to connect the DFT to the amplitudes of various frequencies present in the data.

- The key is the formula for the inverse DFT

$$x_j = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k \, e^{\frac{2\pi i}{N} kj}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k \, e^{(2\pi i k) j/N}$$

$$= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k \, e^{2\pi i k t_j}$$

where $t_j = j/N$ is a sample point

so let $\psi_k(t) = e^{2\pi i k t} = \cos 2\pi k t + i \sin 2\pi k t$

and $\underline{\Psi}(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k \, \psi_k(t)$

the formula above says

$$\underline{\Psi}(t_j) = x_j \quad \text{for } j = 0, \ldots, N-1$$

So $\underline{\Psi}$ interpolates the data exactly

$\boxed{\text{Theorem 2}}$ If $x = x_0, x_1, \ldots$ is a ~~x~~ data stream (time series) sampled at the points $t_0, t_1, \ldots$ with $t_j = j/N$ then

$$\Psi(t) = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \hat{x}_k e^{2\pi i k t}$$

interpolates the data.

- For this reason since power is amplitude squared the collection $|\hat{x}_k|^2$ (or $\frac{1}{N}|x_k|^2$) is called the power spectrum

- Recall $\hat{x}_{N-j} = \overline{\hat{x}_j}$ so $\|\hat{x}_{N-j}\|^2 = \|\hat{x}_j\|^2$

"Conservation of Energy"

$\boxed{\text{Theorem 3}}$ if $\hat{x} = DFT(x)$ then

$$\underbrace{\sum_{k=0}^{N-1} |\hat{x}_k|^2}_{\text{FREQ DOMAIN}} = \underbrace{\sum_{k=0}^{N-1} |x_k|^2}_{\text{Time Domain}}$$

<u>Proof</u> Recall F is unitary and unitary matrices preserve innerproducts and thus norms so

$$\sum_{k=0}^{N-1} |x_k|^2 = \|x\|_2^2 = \|Fx\|_2^2 = \|\hat{x}\|_2^2 = \sum_{k=0}^{N-1} |\hat{x}_k|^2$$

In addition to the relation $|\hat{X}_{N-k}|^2 = |\hat{X}_k|^2$ there is a relation in terms of the associated frequencies evaluated at our sample points $t_j = j/N$

$$\psi_{N-k}(t_j) = e^{2\pi i (N-k) j/N} = e^{2\pi i j}\, e^{-2\pi i k j/N}$$

$$= 1\, \overline{\psi_k(t_j)}$$

so In Theorem 2, we substitute

$$\sum_{k=1}^{N/2} \overline{\hat{X}_k}\, \overline{\psi_k(t)} \quad \text{for} \quad \sum_{k=\frac{N}{2}+1}^{N-1} \hat{X}_k\, \psi_k(t)$$

we get that

since $\psi_0(t) = 1$

$$\eta(t) = \frac{1}{\sqrt{N}}\left[\hat{x}_0 + \sum_{k=1}^{\frac{N}{2}-1}\left(\hat{x}_k\psi_k(t) + \overline{\hat{x}_k\psi_k(t)}\right) + \hat{x}_{\frac{N}{2}}\psi_{\frac{N}{2}}(t)\right]$$

So associated with frequency k; $\psi_k(t) = \cos 2\pi k t + i\sin 2\pi k t$

we have amplitudes

$$|\hat{x}_k|^2 + |\bar{\hat{x}}_k|^2 = 2|\hat{x}_k|^2$$

This leads to the periodogram

$$P_0 = \frac{1}{N} |\hat{X}_0|^2$$

$$P_k = \frac{2}{N} |\hat{X}_k|^2 \qquad k = 1, \ldots, \frac{N}{2} - 1$$

$$P_{\frac{N}{2}} = \frac{1}{N} |\hat{X}_{\frac{N}{2}}|^2$$

[DEMO]

This is the first step in Power spectrum estimation and there any many other considerations

- sampling rate, Nyquist frequencies
- averaging spectrum on overlapping windows
- etc

Now $y(t)$ still has complex functions in it.
To simplify further, notice that for any complex number z, $z + \bar{z} = 2\,\mathrm{Re}(z)$ (Re = real part)

so $$\hat{x}_k \psi_k(t) + \overline{\hat{x}_k \psi_k(t)} = 2\,\mathrm{Re}(\hat{x}_k \psi_k(t))$$

$$= 2\,\mathrm{Re}\left((a_k + i b_k)(\cos 2\pi k t + i \sin 2\pi k t)\right) \quad \text{where } \hat{x}_k = a_k + i b_k$$

$$= 2\left[a_k \cos 2\pi k t - b_k \sin 2\pi k t\right]$$

We also have a term $\hat{x}_{\frac{N}{2}} \psi_{\frac{N}{2}}(t)$

$$= \hat{x}_{\frac{N}{2}}\left(\cos \tfrac{N}{2} 2\pi t + i \sin \tfrac{N}{2} 2\pi t\right).$$ Since the data is real and $\hat{x}_{\frac{N}{2}}$ is real (FACT 1) we just need $\hat{x}_{\frac{N}{2}} \cos(N\pi t)$ for this term. So in the end

<u>Theorem</u>: Given Real DATA $(x_0, x_1, .., x_{N-1})$

Sampled at $(t_0, t_1, .., t_{N-1})$ with $t_j = j/N$ and N even

Then if $\hat{x}_k = a_k + i b_k$

$$T(t) = \frac{1}{\sqrt{N}}\left[\hat{a}_0 + 2\sum_{k=1}^{\frac{N}{2}-1} a_k \cos 2\pi k t - b_k \sin 2\pi k t + \hat{a}_{\frac{N}{2}} \cos N\pi t\right]$$

Interpolates the data, $T(t_j) = x_j$ for $j = 0, .., N-1$

- NOTICE that $T(t)$ is a real function now
- To get the best order M approximation in least squares, you truncate the sum and eliminate the $\frac{N}{2}$-term