Linear Independence

continued → nonzero vectors

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is lin ind if

$$\sum \alpha_i \vec{v}_i = \vec{0} \text{ implies } \alpha_i = 0, \text{ all } i$$

$\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ is lin dependent if

for some α_i not all zero

$$\vec{0} = \sum \alpha_i \vec{v}_i$$

Two vectors - when are they lin. dep

$$\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 = 0 \quad \text{with some } \alpha_i \neq 0$$

$$\vec{v}_1 = -\frac{\alpha_2}{\alpha_1} \vec{v}_2$$

2 vector are lin dep $\iff$ one is a scalar multiple of the other

$$\begin{bmatrix} 3 \\ -6 \end{bmatrix} = 3 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \text{ so } \left\{ \begin{bmatrix} 3 \\ -6 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right\}$$

is lin dep. $\longrightarrow$ spans 1 dim.

3 vector are linearly dependent

$$\alpha_1 \vec{v_1} + \alpha_2 v_2 + \alpha_3 v_3 = 0$$

Some $\alpha_1 \neq 0$.

$$\vec{v_1} = -\frac{\alpha_2}{\alpha_1}\vec{v_2} - \frac{\alpha_3}{\alpha_1}\vec{v_3}$$

So $\vec{v_1}$ is a lin. comb of the other two vectors.

Two cases of lin dep.

$$\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 10 \end{bmatrix}$$

$$2\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 10 \end{bmatrix}$$

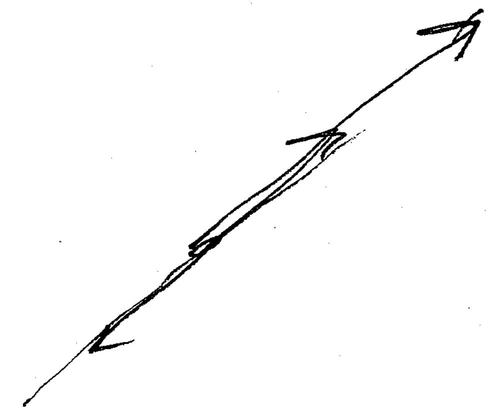
So lin dep, span a plane

= dim 2.

not a basis for $\mathbb{R}^3$.

$2^{\underline{nd}}$ case

$$\underset{\vec{v}_1}{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}} \quad \underset{\vec{v}_2}{\begin{bmatrix} -1 \\ -3 \\ -4 \end{bmatrix}} \quad \underset{\vec{v}_3}{\begin{bmatrix} 2 \\ 6 \\ 8 \end{bmatrix}}$$



$$\vec{v}_2 = -\vec{v}_1 \Rightarrow \text{lin dep,}$$

$$\vec{v}_3 = 2\vec{v}_1$$

~~Span = dim~~

$\dim(\text{span}) = 1.$

$$\text{RANK}\begin{pmatrix} 1 & -1 & 2 \\ 3 & -3 & 6 \\ 4 & -4 & 8 \end{pmatrix} = 1$$

Special case -

$\{\vec{v}_1, \dots, \vec{v}_k\}$ is called an orthogonal set if $\vec{v}_i \perp \vec{v}_j$ if $i \neq j$

FACT orthogonal sets are always lin. ind.

Proof: Assume dep. so

$$\vec{0} = \sum \alpha_i \vec{v}_i$$

say $\alpha_j \neq 0$

$$0 = \vec{v}_j \cdot \vec{0} = \vec{v}_j \cdot \sum \alpha_i \vec{v}_i$$

$$= \sum \alpha_i \vec{v}_j \cdot \vec{v}_i$$

$$= \alpha_j \vec{v}_j \cdot \vec{v}_j \quad \text{Since } \vec{v}_j \cdot \vec{v}_i = 0 \quad i \neq j$$

$$= \alpha_j \|\vec{v}_j\|^2 \neq 0 \quad \text{contradiction.}$$

$$\boxed{\begin{aligned} \vec{w} \cdot \vec{w} &= w_1^2 + w_2^2 + \dots + w_n^2 \\ &= \underbrace{\|\vec{w}\|}_{\substack{\text{magnitude} \\ \text{norm}}}^2 > 0 \text{ if } \vec{w} \neq 0. \end{aligned}}$$

Left vector matrix products

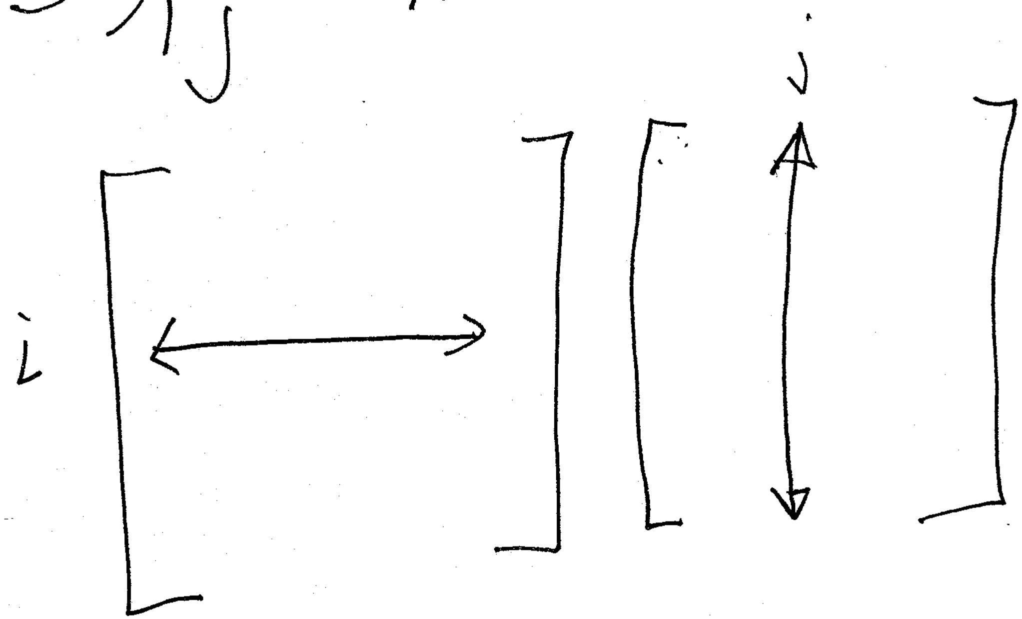
$$\underbrace{\begin{bmatrix} 1 & 2 \end{bmatrix}}_{\text{Row}} \begin{bmatrix} -1 & 4 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 3 & 18 \end{bmatrix}$$

$$(1 \times \cancel{n}) \cdot (\cancel{n} \times m) = (1 \times m)$$

MATRIX-MATRIX MULT.

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 7 & 6 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 11 & \\ & \end{bmatrix}$$

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj}$$



$$(m \times \cancel{n}) \cdot (\cancel{n} \times p) = m \times p$$

$$\boxed{\text{Row} \cdot \text{COL}}$$

$$\begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_m^{\,T} \end{bmatrix} \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_p \end{bmatrix}$$

$$= \begin{array}{c} \\ i \rightarrow \\ \\ \end{array} \begin{bmatrix} \\ \vec{r}_i \cdot \vec{c}_j \\ \\ \end{bmatrix}$$

$$AB = A\left[\vec{c}_1 \,\middle|\, \vec{c}_2 \,\middle|\, \cdots \,\middle|\, \vec{c}_p\right]$$

$$= \left[A\vec{c}_1 \,\middle|\, A\vec{c}_2 \,\middle|\, \cdots \,\middle|\, A\vec{c}_p\right]$$

$\boxed{\text{Outer product}}$

Inner product $\vec{u}^T \vec{v} = \vec{u} \cdot \vec{v}$

$(1 \times n)(n \times 1) = (1 \times 1) = \underline{\underline{\text{Scalar}}}$

$\boxed{\text{Outer product}}$ $\vec{u}\,\vec{v}^T$

$(n \times 1)(1 \times m) = n \times m$ $\underline{\underline{\text{matrix}}}$

example

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} -1 & 3 & 4 \\ -2 & 6 & 8 \\ -3 & 9 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \cdot [-1\ 3\ 4] \\ 2 \cdot [-1\ 3\ 4] \\ 3\,[-1\ 3\ 4] \end{bmatrix}$$

Rowspace has basis $[-1\ 3\ 4]$

so RANK $= 1$

OK

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 3 & 4 \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} -1 \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} & 4 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \end{array} \right]$$

basis for col space is $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

so Col space has dim = 1

so rank = 1

RANK ONE MATRICES ARE FUNDAMENTAL pieces of any matrix — standard method

$$M = R_1 + R_2 + \cdots + R_K$$

all Rank 1.

In large data sets, only small # of R_i needed

$$M \approx R_1 + \cdots + R_j \qquad \boxed{j << K}$$

These decompositions come from product decompositions of M.

These come from treating (matrix)(matrix) products ~~are~~ as (col.) $\circ$ (row)

$$\begin{bmatrix} \vec{c}_1 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} \vec{r}_1^{\,T} \\ \vdots \\ \vec{r}_n^{\,T} \end{bmatrix} = \sum_{i=1}^{n} \underbrace{\vec{c}_i \, \vec{r}_i^{\,T}}_{\text{rank 1 matrix.}}$$

outer products

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} -1 & 3 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 3 \end{bmatrix} \begin{bmatrix} -1 & 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

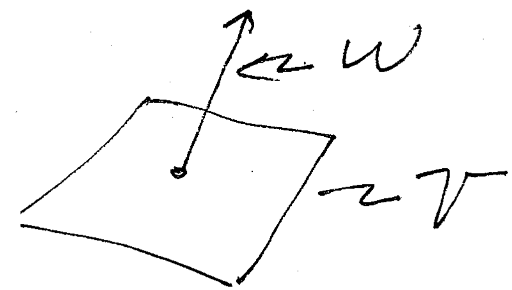
$$= \begin{bmatrix} -1 & 3 \\ -3 & 9 \end{bmatrix} + \begin{bmatrix} 4 & 2 \\ 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix}$$

ORTHOGONALITY and dimension

Say $V \subseteq \mathbb{R}^n$ is a subspace

$W = V^\perp$ is also a subspace



- $V = W^\perp$ orthogonal complement
- $\dim(V) + \dim(W) = n$
- If $\vec{v}_1, \ldots, \vec{v}_k$ is a basis for V and $\vec{w}_1, \ldots, \vec{w}_j$ is a basis for W

$\Rightarrow \{\vec{v}_1 \ldots \vec{v}_k, \vec{w}_1 \ldots \vec{w}_j\}$ is a basis for $\mathbb{R}^n$

called an orthogonal decomposition. $: n$