# Matrices as linear transformations

$T: \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is a linear transformation

if $T(\alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2) = \alpha_1 T(\vec{v}_1) + \alpha_2 T(\vec{v}_2)$

---

Simplest (only) example: $M$ is $m \times n$ matrix

$$T(\vec{x}) = M \vec{x}$$

Matrix ~~is~~ multiplication =
composition of linear transformations

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "T"] \arrow[rr, bend right, "S \circ T"'] & \mathbb{R}^m \arrow[r, "S"] & \mathbb{R}^p \end{tikzcd}$$

$T$ is represented by $M$

$S$ is represented by $N$

$S \circ T$ is represented by $\boxed{NM}$
matrix mult.

Theorem:

FACT: Every choice of basis yields a different matrix representing the linear transformation.

---

**Matrix Inverses**: $M$ is $(n \times n)$ square and there another $N$ $(n \times n)$ with

$$MN = \begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & \ddots \; 1 \end{bmatrix} = I \Rightarrow N \text{ is}$$

called the inverse of $M$ and written $N = M^{-1}$

$$\Rightarrow \quad M^{-1}M = M\,M^{-1} = I$$

Some matrices have inverses

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1}$

$$\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$$

Some matrices don't eg $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

SAY $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$2a + 4c = 1 \rightarrow \boxed{a + 2c = 1/2}$

$\boxed{a + 2c = 0}$ contradiction

If $M$ has an inverse it is called <u>invertible</u> or <u>nonsingular</u>

If $M$ doesn't have an inverse it is called <u>non-invertible</u> or <u>singular</u>.

**Theorem:** $M$ is $n \times n$, The following are equivalent

(1) $M$ is invertible

(2) $\text{rank}(M) = n$

(3) $\text{null}(M) = \{0\}$

(4) $\det(M) \neq 0$

(5) columns of $M$ form a basis for $\mathbb{R}^n$

(6) rows of $M$ form a basis for $\mathbb{R}^n$

(7) . . . . . etc.

# Solving linear equations

$$
m \text{ eq. } \left\{ \begin{array}{l} A_{11} X_1 + \dots \; A_{1n} X_n = b_1 \\ A_{21} X_1 + \dots + A_{2n} X_n = b_2 \\ \quad \vdots \\ A_{m1} X_1 + \dots \; + A_{mn} X_n = b_m \end{array} \right.
$$

$n$ unknowns

More succinctly

$$\boxed{A\vec{x} = \vec{b}}$$

$A$ is $m \times n$
$\vec{x}$ is $n \times 1$
$\vec{b}$ is $m \times 1$

---

1st question: Is there any soln? a unique soln? many soln?

Simplest case $A$ is <u>$n \times n$</u> and <u>invertible</u>

$$A\vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$I\vec{x} = A^{-1}\vec{b}$$

$$\vec{x} = A^{-1}\vec{b}$$

$\boxed{\text{unique soln}}$

General case - geometrically.

$$\vec{b} = A\vec{x} = \begin{bmatrix} \vec{c}_1 & \vec{c}_2 & \cdots & \vec{c}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

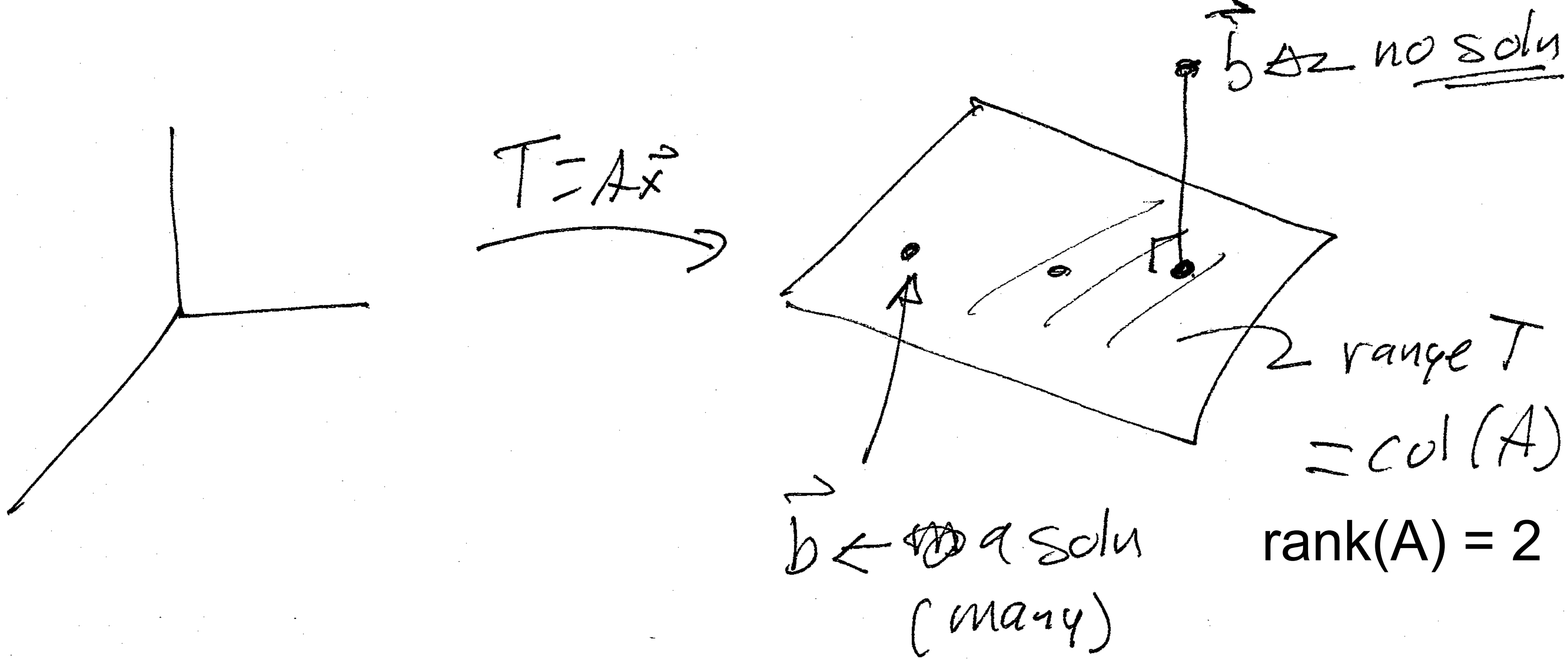
$$= x_1\vec{c}_1 + x_2\vec{c}_2 + \cdots + x_n\vec{c}_n$$

So existence soln is same as whether

$\vec{b}$ is a linear comb. of the col of $A$.

<u>or</u> whether $\vec{b}$ is in the col space of $A$

<u>or</u> $\vec{b}$ is in the range of the linear transformation $T\vec{x} = A\vec{x}$




rank(A) = 2

How do you solve linear equations? — Row reduction = Gausian elimination = <u>LU decomposition</u>

# Moves in row reduction

(1) $C \cdot \mathrm{Row}_i + \mathrm{Row}_j \xrightarrow{\text{put in for}} \mathrm{Row}_j$

(2) swap rows

(3) rescale rows

$C\,\mathrm{Row}_i \longrightarrow \mathrm{Row}_i$

# Row Reduction

$$\underset{A}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}} \xrightarrow[-3R1+R_4 \to R_4]{\substack{-2R1+R_2 \to R_2 \\ -4R_1+R_3 \to R_3}} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 3 & 5 & 5 \\ 0 & 4 & 6 & 8 \end{bmatrix}$$

$$\xrightarrow[-4R2+R_4 \to R_4]{-3R2+R_3 \to R_3} \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 2 & 4 \end{bmatrix} \xrightarrow{-R3+R4 \to R4} \underset{U}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

We will see that this implies

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix} = \underset{L = \text{lower}}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ \underline{2} & 1 & 0 & 0 \\ \underline{4} & 3 & 1 & 0 \\ \underline{3} & 4 & 1 & 1 \end{bmatrix}} \underset{U = \text{upper}}{\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix}}$$

How do we use this to solve

$$LUx = A\vec{x} = \vec{b}$$

$\Big\downarrow$ split into two triangular systems

$$L\vec{y} = \vec{b} \rightsquigarrow \text{has soln } \vec{y}$$

and $U\vec{x} = \vec{y}$

Then $A\vec{x} = LU\vec{x} = L(U\vec{x})$

$= L\vec{y} = \vec{b}$

So $\vec{x}$ solves the system.

$\boxed{\text{How do we use this to solve } A\vec{x}=\vec{b}}$

Secret is that equations like $L\vec{y}=\vec{b}$ and $U\vec{x}=\vec{y}$ are easy to solve by back and forward substitution.

<u>eg</u> Solve First $L\vec{y}=\vec{b}=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}$

$$\begin{bmatrix}1&0&0&0\\2&1&0&0\\4&3&1&0\\3&4&1&1\end{bmatrix}\begin{bmatrix}y_1\\y_2\\y_3\\y_4\end{bmatrix}=\begin{bmatrix}-1\\2\\3\\1\end{bmatrix}$$

as equations

$y_1 = -1$

$2y_1+y_2=2 \longrightarrow y_2=2-2y_1=4$

$4y_1+3y_2+y_3=3 \rightarrow y_3=3-4y_1-3y_2=3+4-12=-5$

$3y_1+4y_2+y_3+y_4=1 \rightarrow y_4=1-3y_1-4y_2-y_3=1+3-16+5=-7$

$\boxed{\text{FORWARD SUBSTITUTION}}$

Now we solve $U\vec{x} = \vec{y}$

$$\begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 11 \\ -22 \end{bmatrix}$$

$$2x_1 + x_2 + x_3 = -1 \qquad x_1 = \ldots$$

$$x_2 + x_3 + x_4 = 4 \rightarrow x_2 = 4 - x_3 - x_4$$

$$2x_3 + 2x_4 = 11 \rightarrow x_3 = \tfrac{1}{2}[11 - 2x_4] = \tfrac{1}{2}[11 + 22]$$

$$2x_4 = -22 \longrightarrow x_4 = -11$$

$\boxed{\text{BACK Substitution}}$

Then $LU(\vec{x}) = L\vec{y} = \vec{b}$ so $\vec{x}$ is sought solution

This is useful to solve many equations

$$\left.\begin{aligned} A\vec{x} &= \vec{b}_1 \\ A\vec{x} &= \vec{b}_2 \\ &\vdots \\ A\vec{x} &= \vec{b}_k \end{aligned}\right\} \text{many eq.}$$

Compute $A = LU$ once and for all and then each equation is fast.

- Sometimes you need to permute rows to compute LU.
- numerical considerations force Row and column swapping (pivots)